



# USA Mathematical Talent Search

## Round 1 Solutions

Year 38 — Academic Year 2026-2027

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**1/1/38.** Place two digits in each square to form a two-digit number. (Leading zeroes are allowed; for example, 01 could be in a square.)

- In each row and column, each digit from 0 to 9 must appear exactly once. All two-digit numbers in the grid must be different.
- A number will be prime if and only if it is circled.
- A given digit in a square must be one of the two digits in that square. (For example, if 8 is given in a square, the square might contain 18 or 81.)
- A number outside a row or column gives the unit digit of the sum of the five numbers in that row or column.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

|   |   |  |   |   |   |
|---|---|--|---|---|---|
|   |   |  | 1 | 5 | 5 |
| 6 | 1 |  | 3 | 2 | 4 |
| 2 |   |  | 0 | 1 | 2 |
| 4 |   |  |   |   | 3 |
| 9 | 4 |  | 8 |   | 8 |

7      4              6

### Solution

|    |    |    |    |    |   |
|----|----|----|----|----|---|
| 08 | 62 | 39 | 71 | 45 | 5 |
| 67 | 19 | 85 | 43 | 20 | 4 |
| 23 | 75 | 64 | 09 | 81 | 2 |
| 14 | 38 | 07 | 25 | 69 | 3 |
| 95 | 40 | 12 | 68 | 73 | 8 |

7      4              6



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**2/1/38.** Alice and Bob start writing down consecutive numbers at the same time, starting with the number “1”. They write numbers at the same rate, meaning that for all  $n \geq 1$ , Alice writes her  $n^{\text{th}}$  number at the same time that Bob writes his  $n^{\text{th}}$  number. Alice writes down Kaktovic numerals, which is a base-20 number system:

$$\underline{1}, \underline{2}, \dots, \underline{19}, \underline{10}, \underline{11}, \dots, \underline{19} \underline{19}, \dots$$

Bob writes down spreadsheet column codes, which follow a base-26 system with no zero:

$$A, B, C, \dots, Z, AA, AB, \dots, ZZ, \dots$$

Whenever Alice and Bob simultaneously write down a number with the same number of digits (in their respective number systems), they earn a *(20-26)-point*.

1. After Alice and Bob have each written down 2026 numbers, how many *(20-26)-points* do they have?
2. Prove that Alice and Bob can only ever earn finitely many *(20-26)-points*, and find the largest value  $n$  such that the  $n^{\text{th}}$  number Alice and Bob each write down earns a *(20-26)-point*.

### Solution

1. Let  $A(n)$  and  $B(n)$  denote the number of digits in the  $n^{\text{th}}$  number Alice and Bob write, respectively. Then we have that for every integer  $k \geq 1$ ,  $A(n) = k$  if and only if

$$20^{k-1} \leq n \leq 20^k - 1.$$

Now we find the range of  $n$  for which  $B(n) = k$ . For fixed  $k$ , the number of  $k$ -digit strings of letters is  $26^k$ . Therefore, the number of strings with fewer than  $k$  letters is

$$26 + 26^2 + \dots + 26^{k-1} = \frac{26^k - 26}{25}.$$

The first  $n$  for which  $B(n) = k$  is then

$$\frac{26^k - 26}{25} + 1 = \frac{26^k - 1}{25}.$$

Similarly, the last  $k$ -digit number in Bob's list is

$$26 + 26^2 + \dots + 26^{k-1} + 26^k = \frac{26^{k+1} - 26}{25}.$$

Therefore,  $B(n) = k$  if and only if

$$\frac{26^k - 1}{25} \leq n \leq \frac{26^{k+1} - 26}{25}.$$

Since  $2026 < 18278 = \frac{26^4 - 26}{25}$  and  $2026 < 7999 = 20^3 - 1$  we only need to look at the intersection of the ranges for  $k = 1, 2, 3$ .



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- $k = 1$ :  $[1, 26] \cap [1, 19] = [1, 19]$  giving us 19 points.
- $k = 2$ :  $[27, 702] \cap [20, 399] = [27, 399]$  giving us 373 points.
- $k = 3$ :  $[703, 18278] \cap [400, 799] \cap [1, 2026] = [703, 2026]$  giving us 1324 points.

Thus, the total  $(20, 26)$ -points Alice and Bob have is  $19 + 373 + 1324 = 1716$ .

2. The largest such  $n$  is

$$n = 20^{12} - 1 = 4,095,999,999,999,999.$$

The existence of a largest point implies that there is some point in which the lower bound in one digit range exceeds the upper bound of the other for some  $k$ . We claim that  $\frac{26^k - 1}{25} > 20^k - 1$  for all  $k \geq 13$ . In particular,  $A(n) \neq B(n)$  for all  $n > 20^k - 1$ . Consider the inequality

$$26^k \geq 25 \cdot 20^k.$$

Dividing both sides by  $20^k$  gives

$$\begin{aligned} \frac{26^k}{20^k} &= \frac{13^k}{10^k} \geq 25 \\ 13^k &\geq 25 \cdot 10^k. \end{aligned}$$

Solving for  $k$ , we get

$$k \geq \frac{\log(25)}{\log(\frac{13}{10})} \approx 12.27.$$

So, since  $k$  is an integer,  $k \geq 13$ . Thus,  $26^k \geq 25 \cdot 20^k$ , meaning

$$\begin{aligned} 26^k &> 25 \cdot 20^k - 24 \\ 26^k - 1 &> 25 \cdot 20^k - 25. \end{aligned}$$

So,

$$\frac{26^k - 1}{25} > 20^k - 1$$

as we claim.

It remains to be checked that  $n = 20^{12} - 1$  satisfies  $A(n) = B(n)$ . Clearly,

$$20^{11} \leq n \leq 20^{12} - 1,$$

so  $A(n) = 12$ . Then,  $20^{12} - 1 < 26^{12} - 1$ , so

$$20^{12} - 1 < \frac{26}{25}(26^{12} - 1)$$



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which is the upper bound for  $B(n) = 12$ . Then,

$$13^{12} = 23,298,085,122,481 < 25,000,000,000,000 = 25 \cdot 10^{12},$$

so

$$2^{12} \cdot 13^{12} \leq 25 \cdot 2^{12} \cdot 10^{12} - 24$$

$$26^{12} \leq 25 \cdot 20^{12} - 24$$

$$26^{12} - 1 \leq 25 \cdot 20^{12} - 25$$

$$\frac{26^{12} - 1}{25} \leq 20^{12} - 1$$

which is the lower bound for  $B(n) = 12$ . So  $A(n) = B(n) = 12$  at  $n = 20^{12} - 1$ , and no larger  $n$  work.

**3/1/38.** There are an infinite number of lily pads, numbered  $0, 1, 2, \dots$  from left to right. A frog is initially on lily pad 0. There exist fixed probabilities  $q_1, q_2, q_3, \dots$ , so that at any point, the probability that the frog leaps  $n$  lily pads to the right is  $q_n$ . (Also,  $q_1 + q_2 + q_3 + \dots = 1$ , so the frog is always leaping to the right.)

It turns out that the probability that the frog lands on lily pad  $n$  is  $p$ , for any positive integer  $n$ .

1. Compute  $q_n$  in terms of  $p$ .
2. Compute the expected number of leaps it takes for the frog to either land on or pass lily pad  $n$ .

## Solution

1. Let  $p_n$  denote the probability that the frog lands on lily pad  $n$ , so  $p_n = p$  for all  $n \geq 1$ . Suppose the frog lands on lily pad  $n$ . Then previously the frog must have been on some other lily pad  $m$  with  $m < n$ . The probability that the frog lands on lily pad  $m$  is  $p_m$ , and it leaps to lily pad  $n$  with probability  $q_{n-m}$ . Summing over  $m < n$ , we get

$$p_n = q_1 p_{n-1} + q_2 p_{n-2} + \dots + q_{n-1} p_1 + q_n.$$

Then,

$$p = q_1 p + q_2 p + \dots + q_{n-1} p + q_n$$

for all  $n \geq 1$ . In the case when  $n = 1$ , we get  $q_1 = p$ .



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Also, we have that for some lily pad  $n + 1$ ,

$$p = q_1p + q_2p + \dots + q_np + q_{n+1}.$$

Comparing these equations, we get

$$pq_n + q_{n+1} = q_n.$$

So,  $q_{n+1} = (1 - p)q_n$ . Therefore,

$$q_n = p(1 - p)^{n-1}, \text{ for all } n \geq 1.$$

2. For a fixed positive integer  $n$ , let  $E_k$  be the expected number of leaps it takes the frog to either land on or pass lily pad  $n$ , when the frog starts on lily pad  $k$ . Then  $E_{n-1} = 1$ .

Suppose the frog is on lily pad  $k$ . Then the probability that it leaps to lily pad  $m < n$  is  $q_{m-k} = p(1 - p)^{m-k-1}$ , and the probability that it leaps to lily pad  $n$  or farther is

$$\begin{aligned} q_{n-k} + q_{n+1-k} + q_{n+2-k} + \dots &= p(1 - p)^{n-k-1} + p(1 - p)^{n-k} + p(1 - p)^{n-k+1} + \dots \\ &= (1 - p)^{n-k-1}(p + p(1 - p) + p(1 - p)^2 + \dots) \\ &= (1 - p)^{n-k-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} E_k &= p(E_{k+1} + 1) + p(1 - p)(E_{k+2} + 1) + \dots + p(1 - p)^{n-k-2}(E_{n-1} + 1) + (1 - p)^{n-k-1} \\ &= pE_{k+1} + p(1 - p)E_{k+2} + \dots + p(1 - p)^{n-k-2}E_{n-1} + 1. \end{aligned}$$

Then the same reasoning with  $E_{k-1}$  gives

$$E_{k-1} = pE_k + p(1 - p)E_{k+1} + \dots + p(1 - p)^{n-k-1}E_{n-1} + 1.$$

From the two equations above,

$$\begin{aligned} E_{k-1} &= pE_k + (1 - p)E_k + p \\ &= E_k + p. \end{aligned}$$

Therefore,

$$E_0 = (n - 1)p + 1.$$



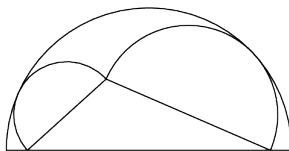
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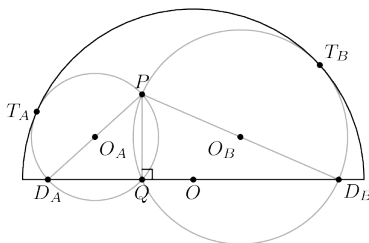
4/1/38. Two semicircles, with radii  $a$  and  $b$ , are inscribed in a semicircle of radius  $r$ , as shown below. Prove that  $a + b = r$ .



## Solution

Let  $\omega$  denote the large semicircle, and let  $\omega_A$  and  $\omega_B$  denote the semicircles with radii  $a$  and  $b$  respectively. Then, let  $\Omega$  denote the full circle of  $\omega$ , and let  $\Omega_A$  and  $\Omega_B$  denote the full circles of  $\omega_A$  and  $\omega_B$  respectively.

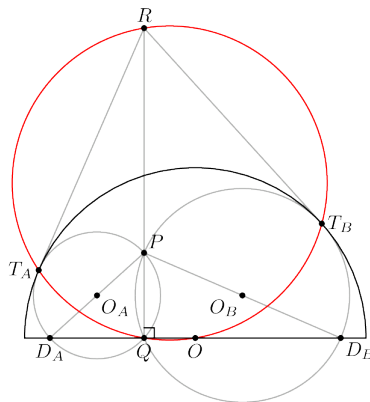
Let  $P$  be the point where  $\omega_A$  and  $\omega_B$  touch, and let  $O$  denote the center of  $\omega$ . Let  $O_A$  denote the center of  $\omega_A$ , and let  $T_A$  and  $D_A$  be the points where  $\omega_A$  touch the arc and diameter of  $\omega$  respectively. Define points  $O_B$ ,  $T_B$  and  $D_B$  similarly.



Let  $Q$  be the projection of  $P$  onto the diameter of  $\omega$ . Since  $\overline{PD_A}$  is a diameter of  $\Omega_A$  and  $\angle PQD_A = 90^\circ$ ,  $Q$  lies on  $\Omega_A$ . Similarly,  $Q$  lies on  $\Omega_B$ .

The radical axis of  $\Omega_A$  and  $\Omega_B$  is  $\overline{PQ}$ . The radical axis of  $\Omega_A$  and  $\Omega$  is their common tangent  $T_A$ . The radical axis of  $\Omega_B$  and  $\Omega$  is their common tangent  $T_B$ . Hence, all three lines concur at  $R$ , the radical center of  $\Omega_A$ ,  $\Omega_B$ , and  $\Omega$ .

Since  $\angle RT_A O = \angle RT_B O = \angle RQO = 90^\circ$ , points  $T_A$ ,  $T_B$ , and  $Q$  lie on the circle with diameter  $\overline{RO}$ .



Since  $\Omega_A$  is tangent to  $\overline{RT_A}$  at  $T_A$ ,  $\angle RT_AP = \angle T_AQP$ .

Since  $RT_AQT_B$  is cyclic,  $\angle T_AQR = \angle T_AT_BR$ .

Since triangle  $RT_AT_B$  is isosceles,  $\angle T_AT_BR = \angle RT_AT_B$ .

Hence,

$$\angle RT_A P = RT_A T_B.$$

so  $P$  lies on  $\overline{T_A T_B}$ .

Since  $\Omega_A$  and  $\Omega$  are tangent at  $T_A$ , there exists a homothety  $h_A$  centered at  $T_A$  taking  $\Omega$  to  $\Omega_A$ . This homothety takes  $T_B$  to  $P$ , so

$$\frac{a}{r} = \frac{T_A P}{T_A T_B}.$$

Similarly,

$$\frac{b}{r} = \frac{T_B P}{T_A T_B},$$

so

$$\frac{a}{r} + \frac{b}{r} = \frac{T_A P}{T_A T_B} + \frac{T_B P}{T_A T_B} = 1.$$

Therefore,  $a + b = r$ .

**5/1/38.** Let  $n \geq 3$  be a positive integer. A circle is marked with  $n$  distinguishable points, and each point is labeled with a positive integer. The sum of all  $n$  labels is  $2n + 3$ , and there is no way to draw an arc on the circle such that the points on the arc have labels summing to  $n + 1$ . In terms of  $n$ , in how many ways can the labels be chosen? (Since the points are distinguishable, rotations and reflections are distinct.)



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For example, if  $n = 3$ , the numbers could be 7, 1, and 1, which gives arcs with sums 0, 1, 2, 7, 8, and 9.

## Solution

The answer is  $\boxed{\frac{n(n+1)(n+2)}{6}}$ .

Let the  $n$  numbers be  $a_1, a_2, \dots, a_n$  and extend the sequence infinitely with  $a_{n+k} = a_k$  for all integers  $k$ . Let  $s_k = \sum_{i=1}^k a_i$  for  $k \geq 0$ , where  $s_0 = 0$ . We are given that  $s_n = 2n + 3$  and  $s_j - s_i \neq n + 1$  for all  $i, j \geq 0$ . This also means  $s_j - s_i \neq n + 2$  for all  $i, j \geq 0$  since this is equivalent to  $s_{n+i} - s_j \neq n + 1$ .

**Claim 1.** At most three of  $a_1, a_2, \dots, a_n$  are greater than 1.

*Proof.* Let  $S$  be the set  $\{s_1, s_2, \dots, s_{n-1}, s_n\}$ . Let  $T_a$  be the set  $\{a, a + 1, a + n + 2\}$  taken as residues mod  $2n + 3$  for  $a = 1, 2, \dots, 2n + 3$ . Every residue is in exactly three of  $T_1, T_2, \dots, T_{2n+3}$ .

Consider the intersection  $T_a \cap S$  for each  $a$ . If  $a + n + 2 \in S$ , then our condition that  $s_j - s_i \notin \{n + 1, n + 2\}$  means that  $a, a + 1 \notin S$ . Therefore, if  $T_a \cap S$  has two elements, they must be both  $a$  and  $a + 1$ .

We have

$$\sum_{i=1}^{2n+3} |T_i \cap S| = 3n,$$

since each of the  $n$  elements of  $S$  is in exactly three of the  $T_a$ . Since  $|T_a \cap S| \leq 2$  for all  $a$ , at least  $3n - (2n + 3) = n - 3$  terms are at least 2. Each such term gives a pair of consecutive residues in  $S$ , which means a distinct  $a_i$  equal to 1. That is,  $n - 3$  of the terms in  $a_1, a_2, \dots, a_n$  are equal to 1.  $\square$

**Claim 2.** There are  $n^2$  sequences with at most two terms greater than 1.

*Proof.* If there is only one term greater than 1, then it must be  $n + 4$  with the remaining terms 1s, which works. There are  $n$  such sequences.

Suppose there are exactly two terms greater than 1, call them  $b_1, b_2$ . Then the arcs that include  $b_1$ , but not  $b_2$  can have any sum  $b_1 + r$  for  $r = 0, 1, 2, \dots, n - 2$ . Similarly, we can obtain any sum  $b_2 + r$ .

In order for  $b_1 + r$  to never equal  $n + 1$  or  $n + 2$ , we require that either  $b_1 \leq 2$  or  $b_1 \geq n + 3$ . The same is required of  $b_2$ . Since  $b_1 + b_2 = n + 5$ , this is only possible if  $\{b_1, b_2\} = \{2, n + 3\}$ . Then where we place  $b_1, b_2$  is arbitrary, because we can never get a contiguous sum equal to either  $n + 1$  or  $n + 2$  when one of the terms is at least  $n + 3$ . There are  $n(n - 1)$  way to choose where the  $n + 3$  and 2 go in the sequence.

In total, we have  $n + n(n - 1) = n^2$  choices.  $\square$



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**Claim 3.** When three terms are greater than one, for each way of choosing the positions of these three terms, there is a unique sequence satisfying the problem conditions.

*Proof.* Let the three numbers greater than 1 in the sequence be  $b_1, b_2, b_3$ . Let  $w_1, w_2, w_3$  be the number of 1s in between the pairs of these numbers, such that  $w_3$  is the count of 1s between  $b_1$  and  $b_2$ ,  $w_1$  is the count of 1s between  $b_2$  and  $b_3$ , and  $w_2$  is the count of 1s between  $b_3$  and  $b_1$ .

Observe that  $b_1 + b_2 + b_3 = n + 6$  and  $w_1 + w_2 + w_3 = n - 3$ . If we draw the arcs containing  $b_1$ , but neither  $b_2$  or  $b_3$ , then these arcs can obtain any sum  $b_1 + r$  for  $0 \leq r \leq w_2 + w_3$ . We need this to never equal  $n + 1$  or  $n + 2$ .

First, observe that  $b_1, b_2, b_3$  are each at most  $n + 2$ . Then, since none can be  $n + 1$  or  $n + 2$  on their own, we have that  $b_1, b_2, b_3$  are each at most  $n$ .

Then, to have  $b_1 + r \notin \{n + 1, n + 2\}$ , we must have that  $b_1 + w_2 + w_3 \leq n$ . Since,  $w_1 + w_2 + w_3 = n - 3$ , this simplifies to  $b_1 - w_1 \leq 3$ . Similarly,  $b_2 - w_2 \leq 3$  and  $b_3 - w_3 \leq 3$ .

Adding these inequalities gives us  $(b_1 + b_2 + b_3) - (w_1 + w_2 + w_3) \leq 9$  so  $(n + 6) - (n - 3) \leq 9$  and we must have that  $b_1 = w_1 + 3$ ,  $b_2 = w_2 + 3$ ,  $b_3 = w_3 + 3$ .

Thus, choosing the positions of  $b_1, b_2, b_3$  in the sequence determines the values of  $w_1, w_2, w_3$  which determines the values of  $b_1, b_2, b_3$ .

Therefore, an arc containing none of  $b_1, b_2, b_3$  will clearly be less than  $n$  and an arc containing exactly one of  $b_1, b_2, b_3$  will sum to at most  $n$ . Any other sequence is the complement of one already analyzed, and since they don't sum to  $n + 1$  or  $n + 2$ , the complements don't either. So, no arc has labels summing to  $n + 1$  or  $n + 2$ .

Thus, there are  $\binom{n}{3} = \frac{n(n-1)(n-2)}{6}$  ways to choose where the three terms are in the sequence. □

Adding the results of our claims together we have

$$n^2 + \frac{n(n-1)(n-2)}{6} = \frac{n(n+1)(n+2)}{6}$$

ways the labels can be chosen.