USA Mathematical Talent Search
Round 3 Solutions
Year 37 — Academic Year 2025–2026
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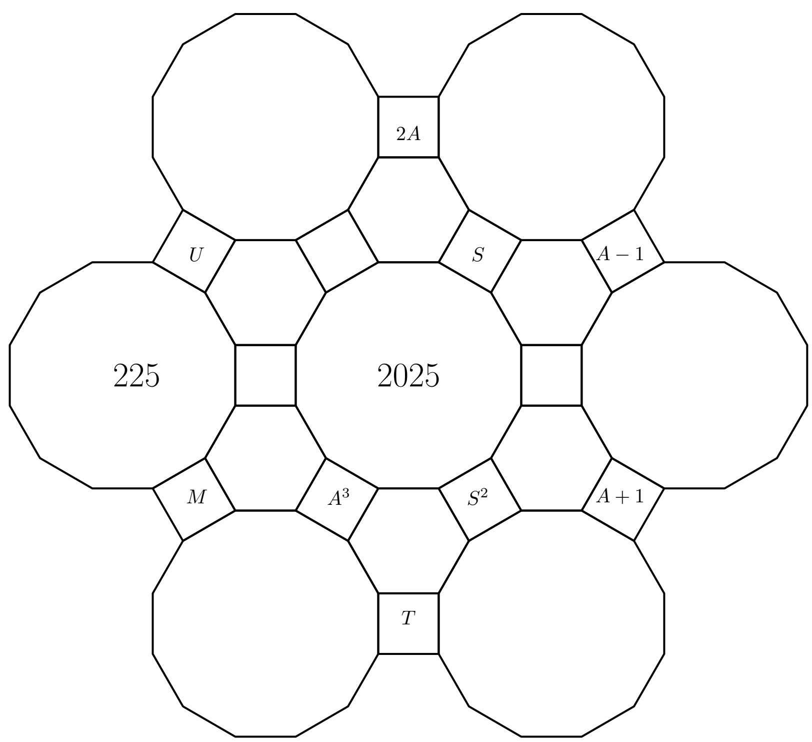
1/3/37. A magical wizard has given you a formidable challenge. Below is a section of a $4-6-12$ tiling of the plane. You must place a positive integer in each square, hexagon, and dodecagon such that:

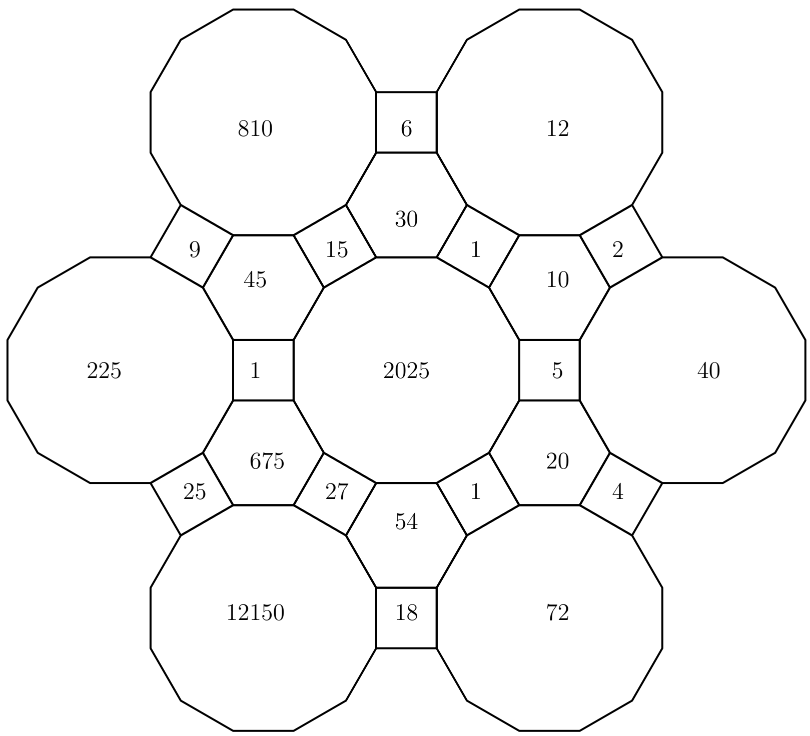
- The only integer that is repeated is 1.
- The value in each dodecagon is the product of the values in the squares connected to it.
- The value in each hexagon is the least common multiple of the values in the three squares connected to it.
- $U + S + A + M + T + S = 57$.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



Solution

The following is the unique solution:



2/3/37. The game of *summing solitaire* is played as follows. A deck of 101 cards (numbered 1 through 101) is randomly shuffled. The cards are drawn one at a time. As each card is drawn, it is put into the *scoring pile* if its number is larger than the numbers on all other cards that have been drawn so far, and otherwise it is discarded. (The first card drawn is placed in the scoring pile.) After drawing all 101 cards, the final score is the sum of all the numbers on the cards in the scoring pile.

What is the probability that the final score is even?

Solution 1:

Instead of 101, suppose there were n cards in the deck. Let a_n denote the probability that the final score is even.

Consider a variant of the game wherein as each card is drawn, it is placed into the scoring pile if it is the *smallest* card drawn so far. Call this variant the *smallest* variant, and call the variant in the original question the *original* variant. For the *smallest* variant of the game, let b_n denote the probability that the final score is even.

Note that if n is odd, then $a_n = b_n$. Indeed, let D_n denote a deck with n cards in some order. Let $S(D_n)$ denote the final score obtained by playing the *original* variant of the game with the deck D_n, and let $S'(D_n)$ denote the final score obtained by playing the *smallest* variant of the game. Let D'_n denote the deck obtained by replacing card i in D_n with card $n+1-i$. Then if n is odd, we see that parity of the cards in D_n are exactly the same as the parity of the cards in D'_n, in order. Also, card i in D_n will be placed into the scoring pile in the *original* variant if and only if card $n+1-i$ in D'_n is placed into the scoring pile in the *smallest* variant. We conclude that $S(D_n)$ has the same parity as $S'(D'_n)$. Since replacing i with $n+1-i$ in a deck gives a bijection from the set of decks to itself, we conclude that the probability that $S(D_n)$ is even is the same as the probability that $S'(D_n)$ is even. In other words, $a_n = b_n$.

We continue by analyzing the *smallest* variant, since 101 is odd. We claim the following expressions for b_n hold for all n:

$$b_n = \begin{cases} \frac{n-2}{2n-2}, & n \text{ even} \\ \\ \frac{n-1}{2n}, & n \text{ odd} \end{cases}$$

We prove these formulas by induction on n. Note that $b_1 = b_2 = 0$, which serve as base cases.

For the induction step, suppose the formulas hold up through $n-1$. Note that if n is even, then $b_n = b_{n-1}$, which proves the formula. This is because card n does not affect the parity of the final score. Next, note that if n is odd, then

$$b_n = \frac{1}{n}(1 - b_{n-1}) + \frac{n-1}{n} \cdot b_{n-1}$$

because card n is scored if and only if it is the first card drawn (with probability $\frac{1}{n}$), and otherwise it is not scored.

By the induction hypothesis, for n odd, we have

$$\begin{aligned} b_n &= \frac{1}{n}(1 - b_{n-1}) + \frac{n-1}{n} b_{n-1} \\ &= \frac{1}{n} \cdot \frac{n-1}{2n-4} + \frac{n-1}{n} \cdot \frac{n-3}{2n-4} \\ &= \frac{n-1}{2n} \end{aligned}$$

which proves the formula for b_n. We conclude that the answer is $\frac{100}{202} = \boxed{\dfrac{50}{101}}$.

Solution 2:

We generalize to a deck with cards labelled 1, 2, …, n.

First, we claim that the probability that the card labelled k is put into the scoring deck is $\frac{1}{n-k+1}$.

To see this, consider the positions of the $n-k+1$ cards labelled k, $k+1$, …, n. Then the card labelled k is put into the scoring deck if and only if k is the first card among all $n-k+1$ positions. The probability that this occurs is $\frac{1}{n-k+1}$.

Now, we want to compute the probability that the sum of all the labels in the scoring deck is even. The cards with even labels have no effect on the parity of the sum, so we can ignore them, leaving the cards with odd labels.

Let p_s be the resulting probability that the sum of the labels is s. Suppose that n is odd. Then the generating function for p_s is

$$\begin{aligned} p(x) &= p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots \\ &= \left(\frac{n-1}{n} + \frac{1}{n}x\right)\left(\frac{n-3}{n-2} + \frac{1}{n-2}x^3\right)\left(\frac{n-5}{n-4} + \frac{1}{n-4}x^5\right)\cdots\left(\frac{2}{3} + \frac{1}{3}x^{n-2}\right)x^n. \end{aligned}$$

Then

$$p_0 + p_1 + p_2 + p_3 + \cdots = p(1) = 1$$

and

$$\begin{aligned} p_0 - p_1 + p_2 - p_3 + \cdots &= p(-1) \\ &= \left(\frac{n-1}{n} - \frac{1}{n}\right)\left(\frac{n-3}{n-2} - \frac{1}{n-2}\right)\left(\frac{n-5}{n-4} - \frac{1}{n-4}\right)\cdots\left(\frac{2}{3} - \frac{1}{3}\right)(-1) \\ &= \frac{n-2}{n}\cdot\frac{n-4}{n-2}\cdot\frac{n-6}{n-4}\cdots\frac{1}{3}\cdot(-1) \\ &= -\frac{1}{n}. \end{aligned}$$

Therefore, the probability that the sum of the labels is even is

$$p_0 + p_2 + \cdots = \frac{1-\frac{1}{n}}{2} = \frac{n-1}{2n}.$$

So for $n = 101$ we get $\frac{100}{202} = \boxed{\frac{50}{101}}$.

3/3/37. Prove that for any positive integer m, there exists a set S of $2m$ consecutive positive integers with the following property: for all nonnegative integers k, n, if S contains one of $\{3^k(3n+1), 3^k(3n+2)\}$, then it also contains the other.

Solution

Let $[a, b]$ denote the set of consecutive positive integers between a and b inclusive. Call such a set good if there is no pair $\{3^k(3n+1), 3^k(3n+2)\}$ for which it contains exactly one element.

Lemma: $[1, 2a]$ is good if a has no 2s in its base 3 representation.

We go by strong induction on a. The base case $a = 1$ is trivial. Now take some $a > 1$ while assuming the hypothesis is true for all smaller a.

Case 1: $a \equiv 1 \pmod 3$. Then $[1, 2a]$ ends in the pair $\{2a-1, 2a\}$. This means $[1, 2a]$ is good if and only if $[1, 2a-2]$ is good. When a in base 3 ends in 1, $a-1$ has no 2s in its base 3 representation if and only if a does, so the induction holds in this case.

Case 2: $a \equiv 2 \pmod 3$. Then a has a 2 in its base 3 representation, so there is nothing to prove.

Case 3: $a \equiv 0 \pmod 3$. Partition $[1, 2a]$ into triples $(3k-2, 3k-1, 3k)$ for $1 \leq k \leq 2a/3$. In each triple, $\{3k-2, 3k-1\}$ form a pair, and $3k$ can be mapped to k in the set $[1, 2a/3]$. This means $[1, 2a]$ is good if and only if $[1, 2a/3]$ is. Likewise, a has no 2s in its base 3 representation if and only if $a/3$ has none, so the induction hypothesis holds.

This completes the induction, and the lemma is proved. ■

Corollary: $[2a+1, 2b]$ is good if both a and b have no 2s in their base 3 representation.

Proof: $[1, 2a]$ and $[1, 2b]$ are both good by the lemma, and their set difference is $[2a+1, 2b]$. The set difference is also good since taking away the elements of a good set will not leave any pair half-filled. ■

Let S be the set of positive integers whose base 3 representation has no 2s. The corollary reduces the problem to showing that for any positive integer m, we can find $a, b \in S$ with $b - a = m$. This is well-known and amounts to showing balanced ternary (base 3 with digits $-1, 0, 1$) can represent all integers, but we present a proof of this here.

For k a nonnegative integer, let T_k be the set of the 3^k integers x with

$$-\frac{3^k-1}{2} \le x \le \frac{3^k-1}{2}.$$

We show by strong induction on k that if $a, b \in S$ with $0 \le a, b < 3^k$, then $b-a$ can take any value of T_k. For the base case $k=0$, $T_k = \{0\}$, and $b-a=0$ when $a=b=0$.

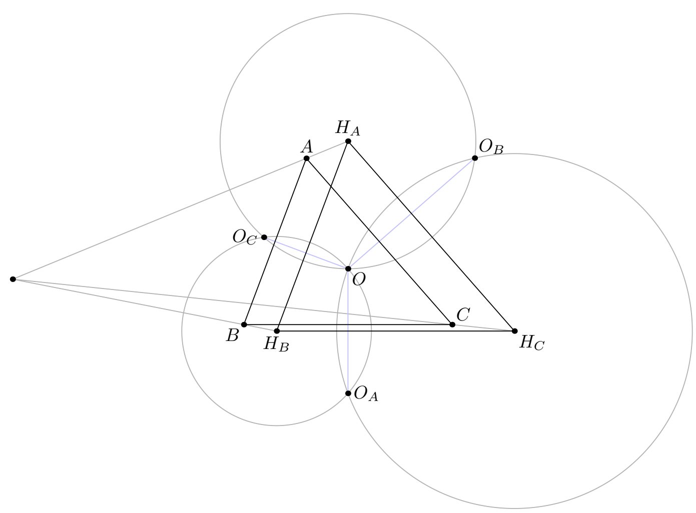
Now assume it is true for k, and consider $a, b \in S$ with $0 \le a, b < 3^{k+1}$. We can write a as a' or $a'+3^k$ for $a' \in S$ and $0 \le a' < 3^k$, and we can do the same for b and b'. Since $0 \le a', b' < 3^k$, the induction hypothesis says $b'-a'$ can be any integer in T_k. Then $b-a$ can be any integer in T_k, any integer in T_k plus 3^k, and any integer in T_k minus 3^k. But this is the same as saying $b-a$ can be any integer in T_{k+1}, so the induction is complete.

4/3/37. Let O be the circumcenter of triangle ABC, and let H be the orthocenter. Let O_A be the circumcenter of triangle BOC, and define O_B, O_C similarly. Let H_A be the circumcenter of triangle OO_BO_C, and define H_B, H_C similarly. Prove that AH_A, BH_B, CH_C, and HO are concurrent.

Solution

Note that H_AH_B is perpendicular to OO_C.

Since $OA = OB$ and $O_CA = O_CB$, the perpendicular bisector of AB is OO_C. Hence, AB is parallel to H_AH_B.



Likewise, AC is parallel to H_AH_C, and BC is parallel to H_BH_C, so triangles ABC and $H_AH_BH_C$ are homothetic. That means AH_A, BH_B, and CH_C concur at the center of homothety.

We have that

$$\angle AOO_B = \frac{1}{2}\angle AOC = B,$$

so $\angle OAO_B = B$. Then $\angle AO_BO = 180^\circ - 2B$.

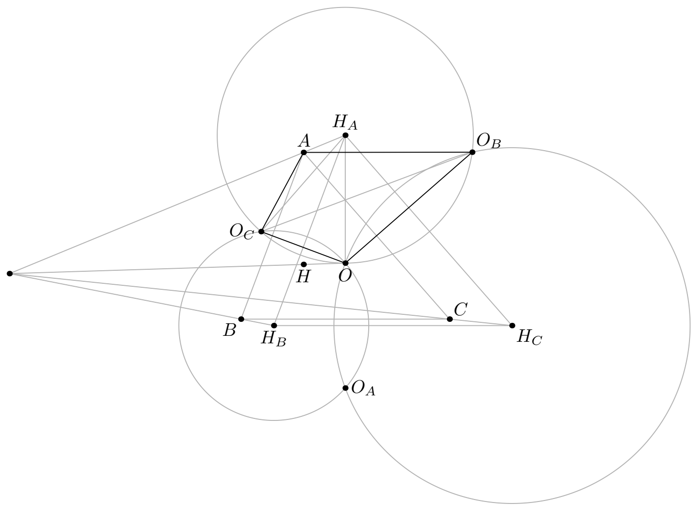
Since AO_BOO_C is a kite,

$$\angle O_CO_BO = \frac{1}{2}\angle AO_BO = 90^\circ - B.$$

Then $\angle O_CH_AO = 2\angle O_CO_BO$, and

$$\angle H_BH_AO = \frac{1}{2}\angle O_CH_AO = 90^\circ - B.$$

That means H_AO is perpendicular to H_BH_C.



Similarly, H_BO is perpendicular to H_AH_C. Hence, O is the orthocenter of triangle $H_AH_BH_C$.

Then H and O are corresponding points in triangles ABC and $H_AH_BH_C$, so HO also passes through the center of homothety.

5/3/37. Let $N > 1$ be an integer, and suppose that $N = \prod_{i=1}^{n} p_i^{e_i}$ is the factorization of N into distinct prime factors $p_1, p_2, \ldots, p_n$. Assume that $p_1 > p_2 > \cdots > p_n$ and define

$$A(N) = \frac{\sum_{i=1}^{n} e_i \cdot p_i}{\sum_{i=1}^{n} e_i}.$$

Further define

$$B(N) = p_1 - \cfrac{1}{p_1 - \cfrac{1}{p_1 - \cfrac{1}{\ddots - \cfrac{1}{p_2 - \cfrac{1}{\ddots - \frac{1}{p_n}}}}}},$$

where p_1 occurs e_1 times in the continued fraction, p_2 occurs e_2 times, and so on. For example,

$$B(3^3 \cdot 5^2) = 5 - \cfrac{1}{5 - \cfrac{1}{3 - \cfrac{1}{3 - \cfrac{1}{3}}}}$$

Find all positive integers N so that $A(N) = B(N)$.

Solution

We claim that $A(N) = B(N)$ if and only if either N is prime or $N = 2^m \cdot 3$ for an integer $m \geq 0$. If $N = p$ is prime, then $A(N) = B(N) = p$ by definition. To show that numbers of the form $2^m \cdot 3$ for positive integers $m \geq 1$ have the desired property, we first need a lemma.

Lemma 1: If $m \geq 1$, then $B(2^m) = \dfrac{m+1}{m}$.

Proof. We prove this by induction on m. For the base case $m = 1$, we have $B(2) = 2 = \dfrac{2}{1}$, hence the two expressions are equal. Suppose that we have proven the claim for all $0 < \ell \leq m$ for some positive integer m. That is, $B(2^\ell) = \dfrac{\ell+1}{\ell}$. If $B(2^{m+1}) = k$ for some rational number k, then $\dfrac{1}{2-k} = B(2^m)$. Since $B(2^m) = \dfrac{m+1}{m}$ by the inductive hypothesis, we find that $2 - k = \dfrac{m}{m+1}$, or

$$k = 2 - \frac{m}{m+1} = \frac{m+2}{m+1}.$$

This completes the inductive step, hence the lemma is proven. ■

If $N = 2^m \cdot 3$ for any integer $m \geq 1$, then $B(N) = 3 - \dfrac{1}{B(2^m)}$. By Lemma 1 we then have

$$B(2^m \cdot 3) = 3 - \frac{m}{m+1} = \frac{2m+3}{m+1}.$$

Since we also have $A(2^m \cdot 3) = \dfrac{2m+3}{m+1}$ by definition, we have proven that $A(N) = B(N)$ whenever $N = 2^m \cdot 3$ for an integer $m \geq 1$.

Next we prove the converse, that is if $A(N) = B(N)$ then either N is prime or $N = 2^m \cdot 3$ for some positive integer m. Suppose N is not prime and write $N = \prod_{i=1}^{n} p_i^{e_i}$, where the p_i are distinct prime factors and $n \geq 2$. Assume WLOG that p_1 is the largest prime and p_2 is the second-largest prime distinct from p_1. In general, the constructions of $A(N)$ and $B(N)$ make sense for arbitrary ordered finite sequences of integers $\mathbf{x} = (x_1, x_2, \cdots x_n)$.

Lemma 2. Suppose $B(\mathbf{x}) = \frac{N(\mathbf{x})}{D(\mathbf{x})}$ with $\gcd(N(\mathbf{x}), D(\mathbf{x})) = 1$, then

- $N(\mathbf{x}) \geq D(\mathbf{x}) + 1$, with equality if and only if $x_1 = x_2 = \cdots = x_n = 2$;
- $D(\mathbf{x}) \geq n$, with equality if and only if $x_2 = x_3 = \cdots = x_n = 2$.

Proof. We will prove this by induction on the length of a decending list of primes $\mathbf{x} = (x_1, x_2, \cdots, x_n)$. Our base case is $B((x_1)) = x_1 = \dfrac{x_1}{1}$. Then $N((x_1)) = x_1 \geq 2$ and $D(x) = 1$, so $N((x_1)) \geq D((x_1)) + 1$. Moreover, $N((x_1)) = 2$ if and only if $x_1 = 2$. The statement about $D(x)$ is vacuously true. Now suppose the statement in the lemma is true for all descending lists of primes of length $m < n$ for some $n \geq 2$, and let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ be a list of length n. By definition of $B(\mathbf{x})$, we have

$$B(x_1, x_2, \ldots, x_n) = x_1 - \frac{1}{B(x_2, \ldots, x_n)}.$$

Let $\mathbf{x}' = (x_2, x_3, \cdots, x_n)$ and write $B(\mathbf{x}')$ as a fraction $\dfrac{N(\mathbf{x}')}{D(\mathbf{x}')}$ in lowest terms. By the inductive hypothesis, we know that:

- $N(\mathbf{x}') \geq D(\mathbf{x}') + 1$, with equality if and only if $x_2 = x_3 = \cdots = x_n = 2$ (note the respect of the indices in $\mathbf{x}'$).
- $D(\mathbf{x}') \geq n - 1$, with equality if and only if $x_3 = x_4 = \cdots = x_n = 2$.

Suppose we write $B(\mathbf{x})$ as a fraction $\dfrac{N(\mathbf{x})}{D(\mathbf{x})}$ in lowest terms. We have

$$\begin{aligned}\frac{N(\mathbf{x})}{D(\mathbf{x})} &= x_1 - \frac{1}{B(x_2,\ldots,x_n)} \\ &\geq x_1 - \frac{n-1}{n}\end{aligned}$$

since $D(\mathbf{x}') \geq n-1$ and $N(\mathbf{x}') \geq D(x\mathbf{x}')+1 \geq n$. Since we also know that $x_1 \geq 2$, we deduce that

$$\begin{aligned}N(\mathbf{x}) &\geq D(\mathbf{x})\left(x_1 - (1-\frac{1}{n})\right) \\ &\geq D(\mathbf{x})\left(2 - (1-\frac{1}{n})\right) \\ &= D(\mathbf{x}) + \frac{D(\mathbf{x})}{n}.\end{aligned}$$

Hence if we can show that $D(\mathbf{x}) \geq n$, then we get $N(\mathbf{x}) \geq D(\mathbf{x})+1$ as well. Towards that end, starting at $B(\mathbf{x}) = x_1 - \dfrac{D(\mathbf{x}')}{N(\mathbf{x}')}$, note that the smallest $N(\mathbf{x}')$ can be is n, which occurs if and only if $x_2 = x_3 = \cdots = x_n = 2$. In general, since $N(\mathbf{x}') \geq D(\mathbf{x}')+1$, the fraction

$$\frac{N(\mathbf{x})}{D(\mathbf{x})} = x_1 - \frac{D(\mathbf{x}')}{N(\mathbf{x}')} = \frac{N(\mathbf{x}')x_1 - D(\mathbf{x}')}{N(\mathbf{x}')}$$

cannot be reduced further, so $D(\mathbf{x}) \geq N(\mathbf{x}') \geq n$. Thus we have proven $D(\mathbf{x}) \geq n$ and $N(\mathbf{x}) \geq D(\mathbf{x})+1$. Moreover, using the inductive hypothesis again, it's apparent that these inequalities for $D(\mathbf{x})$ and $N(\mathbf{x})$ are equalities if and only if $x_2 = x_3 = \cdots = x_1 = 2$ and $x_1 = x_2 = \cdots = x_n = 2$, respectively. This completes the inductive step. ■

Now we can prove the original claim that $A(N) = B(N)$ if and only if N is prime or $N = 2^m \cdot 3$ for any $m \geq 1$. Indeed if $N = \prod_{i=1}^n p_i^{e_i}$, then the denominator of $A(N)$ is at most n when written as a fraction in lowest terms. On the other hand, the lemma shows us the denominator of $B(N)$ is at least n when written as a fraction in lowest terms, with equality if and only if $n = 2$, $e_1 = 1$, and $p_2 = 2$. Let $m = e_2$ so the $N = p_1 \cdot 2^m$. Then we have

$$A(N) = \frac{p_1 + 2m}{m+1},$$

and from Lemma 1 we can compute

$$B(N) = p_1 - \frac{1}{B(2^m)} = p_1 - \frac{m}{m+1} = \frac{(m+1)p_1 - m}{m+1}.$$

Thus we have $p_1 + 2m = (m+1)p_1 - m$, or $3m = p_1 m$. If follows that $p_1 = 3$, and the proof is complete. ■

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