USA Mathematical Talent Search
Round 1 Solutions
Year 37 — Academic Year 2025–2026
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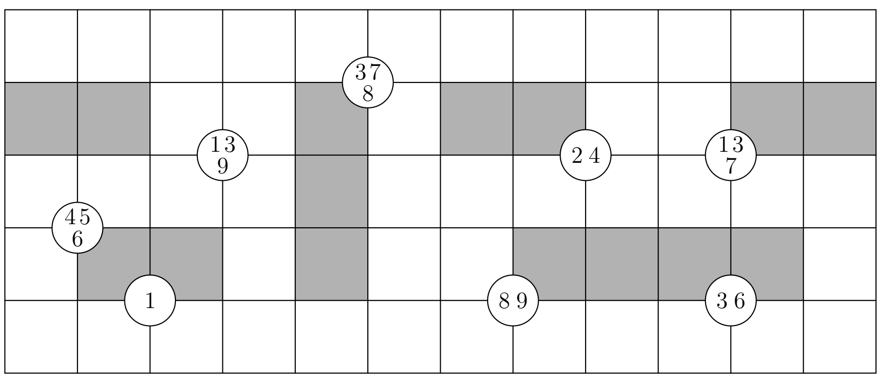
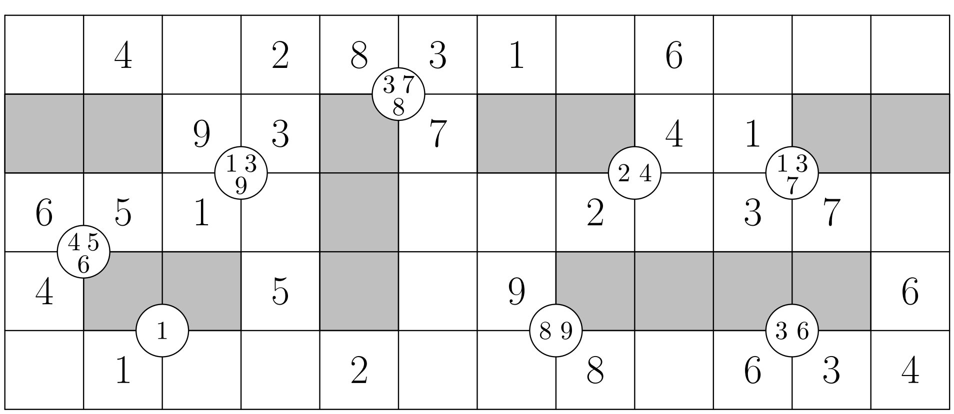
1/1/37. Place a nonzero digit into some of the white cells of the grid. Shaded cells must remain blank. No digit can repeat in a row or column. In each row, the sum of the digits must be equal to some fixed value R (which you find during the solution). Similarly, in each column, the sum of the digits must be equal to some fixed value C. Circles in the grid give all the digits in the cells that touch the circle. (Including repeats; if two cells touching a circle have the same digit, the circle must contain that digit twice.) Some of the digits you place may not be adjacent to one of the circles, but every digit in the circles must be used in an adjacent white cell.

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)



Solution



2/1/37. When Nia was learning about decimals, her teacher asked her to add two terminating decimals with a positive integer part. Then, Nia's teacher asked her to multiply those same two decimals. To Nia's surprise, the (correct) result of both computations was the same!

Both of the two numbers Nia's teacher gave her were non-whole positive numbers. When written as decimals without trailing zeroes, both of these numbers also had the same number of digits after the decimal point, and did not contain the digit 0. Find all possible pairs of numbers Nia's teacher could have given her.

Solution

Suppose the two numbers Nia's teacher gave her are x and y. Since x and y have the same number of digits after the decimal point, we know there is a positive integer k such that $10^k \cdot x$ and $10^k \cdot y$ are both integers.

Suppose that $a = 10^k \cdot x$ and $b = 10^k \cdot y$. We know that $x + y = xy$, so it follows that $\frac{a}{10^k} + \frac{b}{10^k} = \frac{ab}{10^{2k}}$. Simon's Favorite Factoring Trick then tells us that

$$(a - 10^k)(b - 10^k) = 10^{2k}.$$

If either $a - 10^k$ or $b - 10^k$ are negative, both quantities must be negative, which would imply that both x and y are negative, a contradiction. So both quantities are positive; hence $a > 10^k$ and $b > 10^k$, so $x > 1$ and $y > 1$. This means that a and b have a zero digit if and only if x and y do; hence neither a nor b contains a 0 in their decimal representations. In particular, this tells us that neither a nor b is a multiple of 10, so WLOG we must have $a - 10^k = 2^{2k}$ and $b - 10^k = 5^{2k}$. This means that

$$x = \frac{10^k + 2^{2k}}{10^k} = 1 + \left(\frac{2}{5}\right)^k,$$
$$y = \frac{10^k + 5^{2k}}{10^k} = 1 + \left(\frac{5}{2}\right)^k.$$

If $k = 1$, this gives us the solution $\boxed{(1.4, 3.5)}$. If $k = 2$, this gives us the solution $\boxed{(1.16, 7.25)}$. Both of these work since

$$\begin{aligned} 1.4 + 3.5 &= 4.9 = 1.4 \cdot 3.5, \\ 1.16 + 7.25 &= 8.41 = 1.16 \cdot 7.25. \end{aligned}$$

On the other hand, when $k \geq 3$, we find that $x = 1 + \left(\frac{2}{5}\right)^k$ satisfies $1 < x < 1.1$, so all these values of x have a 0 in the tenths place. So these are the only solutions.

3/1/37. Let (a, b) be a pair of rational numbers. Every minute, we are allowed to modify the pair in one of the following ways:

i. Replace (a, b) with $(a+1, b+1)$

ii. If $a \neq 0$ and $b \neq 0$, replace (a, b) with $\left(\frac{1}{a}, \frac{1}{b}\right)$

iii. Replace a with $-a$

iv. Replace b with $-b$

(a) Suppose we start with the pair $(0, 0)$. Is it possible to modify this pair to eventually equal $\left(2025, \frac{1}{2025}\right)$?

(b) Suppose we start with the pair $(0, 1)$. Is it possible to modify this pair to eventually equal $\left(2025, \frac{1}{2025}\right)$?

Solution

(a) Yes, it is possible. We start with a lemma.

Lemma. If (a, b) is attainable, then $(a, b+2)$ and $(a+2, b)$ are attainable.

Proof. Start with the pair (a, b). Then perform modifications (i), (iii), (i), (iii) as follows:

$$(a, b) \xrightarrow{\text{(i)}} (a+1, b+1) \xrightarrow{\text{(iii)}} (-a-1, b+1) \xrightarrow{\text{(i)}} (-a, b+2) \xrightarrow{\text{(iii)}} (a, b+2).$$

Similarly, we can perform modifications (i), (iv), (i), (iv) in that order to obtain $(a+2, b)$. □

We proceed by first applying modification (i) to obtain $(1, 1)$, and then applying the lemma 1012 times to obtain $(1, 2025)$.

Next, we apply modification (ii) to obtain $\left(1, \frac{1}{2025}\right)$. We then apply the lemma another 1012 times to obtain $\left(2025, \frac{1}{2025}\right)$.

(b) No, it is not possible. For a given pair $\left(\frac{a_1}{a_2}, \frac{b_1}{b_2}\right)$, we consider the parity of the value $|a_1b_2 - a_2b_1|$ (where we assume that (a_1, a_2) are not both even, and (b_1, b_2) are not both even). We claim that this parity is unchanged by all the given operations. We can check this one at a time as follows:

1. Starting with $\left(\frac{a_1}{a_2} + 1, \frac{b_1}{b_2} + 1\right) = \left(\frac{a_1+a_2}{a_2}, \frac{b_1+b_2}{b_2}\right)$, we have

$$|b_2(a_1 + a_2) - a_2(b_1 + b_2)| = |a_1b_2 - b_1a_2|.$$

So in this case, the value itself is unchanged.

2. Given the pair of reciprocals $\left(\frac{a_2}{a_1}, \frac{b_2}{b_1}\right)$, we have

$$|a_2b_1 - a_1b_2| = |a_1b_2 - a_2b_1|.$$

So the value is unchanged again in this case.

3. For the pair $\left(\frac{-a_1}{a_2}, \frac{b_1}{b_2}\right)$, we have

$$|-a_1b_2 - a_2b_1| = |a_1b_2 - a_2b_1 - 2a_1b_2|,$$

which has the same parity as $|a_1b_2 - a_2b_1|$. Note that this parity is also the same if we negate a_2 instead of a_1.

4. Similarly, for the pair $\left(\frac{a_1}{a_2}, \frac{-b_1}{b_2}\right)$, we have

$$|a_1b_2 + a_2b_1| = |a_1b_2 - a_2b_1 + 2a_2b_1|,$$

where the parity is again unchanged.

To conclude, note that $|0 \cdot 1 - 1 \cdot 1| = 1$ is odd, whereas $|2025 \cdot 2025 - 1 \cdot 1|$ is even. So it is impossible to modify $(0, 1)$ into $\left(2025, \frac{1}{2025}\right)$.

4/1/37. Let S be a finite set of points in the plane such that no three points in S are collinear. Suppose that there are two triangles whose vertices are six distinct points in S, such that their intersection is a hexagon with no points of S in its interior or on its boundary. Prove that there is a convex hexagon with vertices in S that has no points of S in its interior.

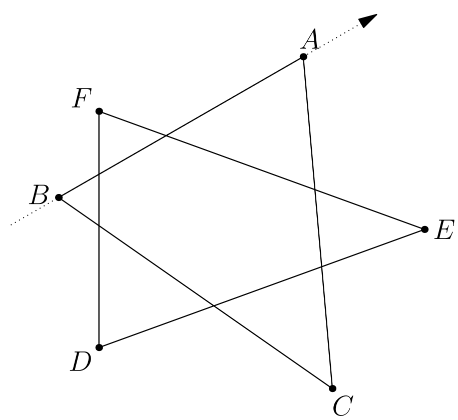
Solution

First we prove a lemma.

Lemma. If the intersection of triangles ABC and DEF is a hexagon with no points of S in its interior, then $\{A, B, C, D, E, F\}$ are the vertices of a convex hexagon: in particular, the vertices of triangles ABC and the vertices of DEF can each be permuted such that $AFBDCE$ is a convex hexagon.

Proof. The sides of the polygon formed by the intersection of triangles ABC and DEF are subsegments of the sides of triangles ABC and DEF, but since there are only six sides among triangles ABC and DEF, all of the sides among triangles ABC and DEF are used. Therefore, each side of ABC is intersected by two sides of DEF, and vice versa.

Permute the vertices of triangles ABC and DEF so that each pair of sides (AB, DF), (AB, FE), (BC, DE), (BC, FD), (CA, DE), and (CA, EF) intersect.



Since segment $\overline{AB}$ intersects $\overline{DF}$ and $\overline{FE}$, F and $\{D, E\}$ are on opposite sides of line AB. If C and F were on the same side as line AB, then C and $\{D, E\}$ would be on opposite sides of line AB, so segments $\overline{AC}$ and $\overline{DE}$ would not intersect, which is a contradiction. Therefore, F and C are on opposite sides of line AB, so triangles AFB and ABC do not intersect.

By a similar argument, E and B are on opposite sides of line AC, so triangles AEC and ABC do not intersect. Since F and $\{C, E\}$ are on opposite sides of line AB, triangles AFB

and AEC do not intersect. Therefore, triangles AFB, ABC, and AEC do not intersect.

Similarly, we can prove that triangle BDC does not intersect the triangles AFB, ABC, or AEC, so the triangles AFB, AEC, BDC, and ABC do not intersect. We conclude that hexagon $AFBDCE$ is non-self-intersecting.

Then, since $AFBD$, $FBDC$, $BDCE$, $DCEA$, $CEAF$, and $EAFB$ are convex quadrilaterals, each angle of the non-self-intersecting polygon $AFBDCE$ is convex, so $AFBDCE$ is a convex hexagon, as desired.

□

Now consider all 6-tuples (A, B, C, D, E, F) of points in S such that the intersection of triangles ABC and DEF is a hexagon with no points of S in its interior, and $AFBDCE$ is a convex hexagon. By the given information and the previous lemma, there exists at least one such tuple. Select a tuple (A, B, C, D, E, F) which minimizes the size of set T, the points of S in the interior of hexagon $AFBDCE$. We will show that $T = \emptyset$, which would demonstrate that $AFBDCE$ has no points of S in its interior.

First, observe that each point of S is either in the interior or exterior of any polygon with points in S due to the non-collinearity condition. If AFB contains some point $X \in S$ in its interior, then the set of points of S in the polygon $AXBDCE$ is a strict subset of T since $AXBDCE$ is inside $AFBDCE$ and X is not in the interior of $AXBDCE$, but this contradicts the minimality of the size of the set T. By similar logic, the triangles BDC, CEA, DCE, EAF, and FBD also do not contain any points of S in their interiors.

Since $AFBDCE$ is composed of triangles AFB, BDC, CEA, and ABC, each point in T is in the interior of ABC. Similarly, since $AFBDCE$ is composed of triangles DCE, EAF, FBD, and DEF, each point in T is in the interior of DEF. Therefore, each point in T is in the interior of the intersection of ABC and DEF. However, since there are no points in the interior of the intersection of ABC and DEF, $T = \emptyset$, as desired.

5/1/37. Let n be a positive integer. Call a coloring of a rectangular grid k-*good* if

- Each cell of the grid is colored with one of n colors,
- There are the same number of cells of each color, and
- Every row and column has at least k cells of the same color. (The color with at least k cells can vary across rows and columns, e.g., if two of the colors are red and blue, then it's possible for the first row to have at least k red cells, while the second row has at least k blue cells.)

For all n, find the largest positive integer k (which may or may not depend on n) such that there exists a k-good coloring of an $n^2 \times n^2$ grid.

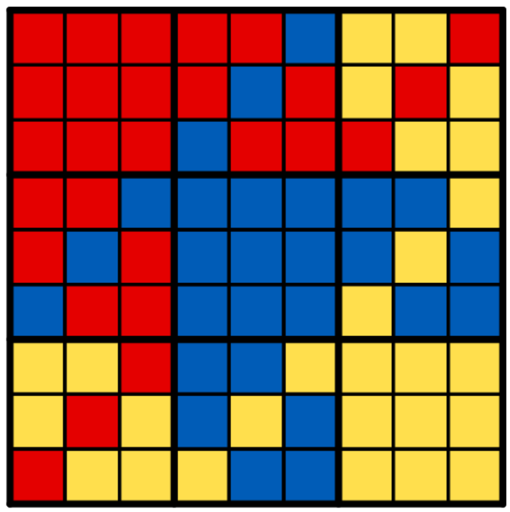
Solution

Answer: $k = \dfrac{n(n+1)}{2}$.

Construction: Given any n, we will first show that there is a k-good coloring of an $n^2 \times n^2$ grid with $k = \dfrac{n(n+1)}{2}$. Number the colors from 1 through n, and divide the $n^2 \times n^2$ grid into n^2 non-overlapping $n \times n$ subgrids. We will call these subgrids n-blocks for clarity. In particular, our $n^2 \times n^2$ grid is built from n^2 non-overlapping n-blocks, and in the rest of the construction we may view the original grid as also a $n \times n$ grid made up of the n-blocks. We will use capital indices I and J to indicate the position of each n-block, e.g. the n-blocks along the diagonal are at positions $(I, J) = (1, 1), (2, 2), \ldots, (n, n)$. In general, we refer to these positions as block-rows and block-columns to distinguish it from the original $n^2 \times n^2$ grid of rows and columns. Let $a_1, \ldots, a_n$ be symbols. For each n-block, place the symbols in the cells of the n-block so that all n symbols appear in each row and column of the n-block. For example, with $n = 3$, the symbols are a_1, a_2, and a_3, and an n-block could look like:

a_1	a_2	a_3
a_2	a_3	a_1
a_3	a_1	a_2

We will use this auxiliary coloring of the n-blocks to guide our construction of a $\dfrac{n(n+1)}{2}$-good coloring. (Note: these auxiliary colorings of the n-blocks are *Latin Squares.*) With reference to the $n \times n$ grid of n-blocks, for each $1 \le I \le n$, color the n-block in the I^{th} block-row and block-column entirely with color I. So the n-blocks along the diagonal are each a different color. Then for all $1 \le J \le n-1$, consider the two n-blocks located in the I^{th} block-row and $(I+J)^{th}$ block-column and the $(I+J)^{th}$ block-row and I^{th} block-column (considering indices cyclically). For each of these two n-blocks, take all the cells with $n - J$ distinct auxiliary symbols and color them with color I, in such a way that doesn't overlap with other cells that we've already colored. For example, if $n = 3$ and $I = J = 1$, this construction says color the 3-block in the top-left corner of the block-grid entirely with color 1. Then in the 3-blocks located at block-indices $(1, 2)$ and $(2, 1)$, first pick $3 - 1 = 2$ of the symbols a_1, a_2, a_3, and then color all of the cells with those symbols using color 1. Here's what this coloring looks like for $n = 3$, where our three colors are red, blue, and yellow in that order:



We can verify this does result in a valid coloring of the grid. Note that the cells within each n-block are only colored with two colors, which correspond to the block-row and block-column that the n-block is located in. Then, if an n-block is located in block-row I or block-column I with respect to the block-grid, a total of $n + (n-1) + \cdots + 1 = \dfrac{n(n+1)}{2}$ cells in each row and column of the main grid overlapping with that n-block are colored with color I, so this is an $\dfrac{n(n+1)}{2}$-good coloring of an $n^2 \times n^2$ grid.

Bound: Take a k-good coloring of an $n^2 \times n^2$ grid, and suppose one of the n colors used in this coloring is red. For any color c, call a row or column in this grid a *c-line* if at least k cells in that row or column are colored c. By hypothesis, every row and column is a c-line for some color c. But since there are n colors, n^2 rows, and n^2 columns, the Pigeonhole Principle implies there is some color c such such at least $2n$ rows and columns are c-lines. We can suppose without loss of generality that there are at least $2n$ red-lines. So if r is the number of red rows and c is the number of red columns in this grid, then $r + c \geq 2n$.

Now, since each red row and column has at least k red cells, there are at most $r(n^2 - k)$ non-red cells in the red rows of our grid, and at most $c(n^2 - k)$ non-red cells in the red columns of our grid. If a non-red cell is not in a red row or column of our grid, it must lie in the intersection of the $(n^2 - r)$ non-red rows of our grid and the $(n^2 - c)$ non-red columns of our grid. This means that the number of non-red cells in our grid is at most

$(n^2-k)(r+c)+(n^2-r)(n^2-c)$. Since there are exactly n^4-n^3 non-red cells in our grid, it follows that

$$n^4-n^3 \le (r+c)(n^2-k)+(n^2-r)(n^2-c).$$

Simplifying, this gives us $(r+c)k-n^3 \le rc$. By AM-GM, we know that $\frac{(r+c)^2}{4} \ge rc$, which gives us

$$(r+c)^2-4k(r+c)+4n^3 \ge 0.$$

Completing the square, this gives us

$$((r+c)-2k)^2 \ge 4k^2-4n^3.$$

Since $r+c \ge 2n$, we know that $2k-2n \ge 2k-(r+c)$, so

$$(2k-2n)^2 = (2n-2k)^2 \ge ((r+c)-2k)^2.$$

Hence, we get

$$(2k-2n)^2 \ge 4k^2-4n^3,$$

which simplifies to $k \le \frac{n(n+1)}{2}$.