USA Mathematical Talent Search

Round 3 Solutions

Year 36 — Academic Year 2024–2025

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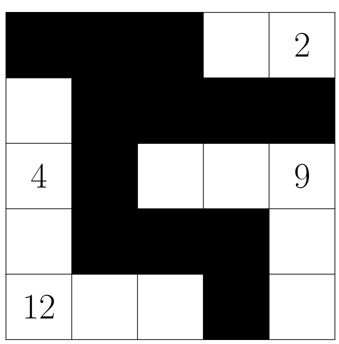
1/3/36. Shade some squares in the grid so that:

1. Squares with numbers are unshaded.
2. Each number is equal to the product of the number of unshaded squares it can "see" in its row and column. (A square can see another square if they're in the same row or column and the sight line between them doesn't have any shaded squares. Each square can see itself.)
3. The shaded squares must make one connected group. Two squares are considered to be connected if they share an edge.

									4
			6			6			
		36	24						
				16					
					24				
						36	18		
			6			12			
36									

The following is an example of a completed puzzle to clarify the rules.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

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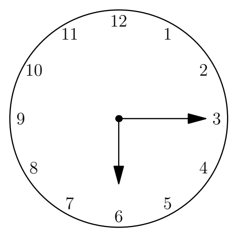
Solution

■	■	■	■	■	■	■	■	■	4
	■		6			6		■	
	■		■	■	■	■	■	■	
	■	36	24					■	
	■			16		■	■	■	■
	■				24			■	■
						36	18		■
	■	■	■	■	■		■	■	■
	■		6		■	12			■
36				■	■	■	■	■	■

2/3/36. Calamitous Clod deceives the math beasts by changing a clock at Beast Academy. First, he removes both the minute and hour hands, then places each of them back in a random position, chosen uniformly along the circle.

Professor Grok notices that the clock is not displaying a valid time. That is, the hour and minute hands are pointing in an orientation that a real clock would never display. One such example is the hour hand pointed at 6 and the minute hand pointed at 3.



The math beasts can fix this, though. They can turn both hands by the same number of degrees clockwise. On average, what is the minimal number of degrees they must turn the hands so that they display a valid time?*

**Assume that after Calamitous Clod replaces the hands, they don't move again until the math beasts adjust their position.*

Solution

Consider the set of pairs of angles (m, h), $0° \leq m, h < 360°$. Each pair corresponds to a clock position (possibly invalid), where m and h determine the number of degrees clockwise from pointing at 12. For example, $(180°, 195°)$ would be the pair corresponding to 6:30. Let S be the set of pairs that show a valid clock position.

Let the angle between the hands after Calamitous Clod replaces them be θ. Notice that, relative to the hour hand, the minute hand travels at a speed of $6° - \frac{1}{2}° = \frac{11}{2}°$/min. Thus, it always takes

$$\frac{360°}{\frac{11}{2}°/\text{min}} = \frac{720}{11}\text{ min}$$

before the minute and hour hands have the same angle between them. In other words, if (m_1, h_1) and (m_2, h_2) correspond to times $\frac{720}{11}$ minutes apart, then $|m_1 - h_1| = |m_2 - h_2|$.

The display of an analog clock repeats every 12 hours. Thus, it suffices to consider a 720-minute period. Since the angle between hands repeats every $\frac{720}{11}$ minutes, this implies that for any $0° \leq \theta \leq 180°$, there are exactly 11 pairs in S whose angles have a difference of θ. Furthermore, the values for the minute hand in those 11 pairs are equally spaced, since between pairs it travels

$$\frac{720}{11} \cdot 6° = \frac{4320°}{11}.$$

After subtracting $360°$, we get $\frac{360°}{11}$. Thus, the 11 values for m are equally spaced along the interval $[0°, 360°)$.

Since the angle is fixed once Calamitous Clod sets the hands, it is sufficient to consider the position of the minute hand. There are exactly 11 valid positions for the minute hand, which are equally spaced along the circle. Thus, the minute hand always lands in an interval between two valid positions. This is equivalent to uniformly selecting a random angle α on the interval $[0°, \frac{360°}{11})$. The minimal number of degrees the math beasts must turn the hands is

$$\frac{360°}{11} - \alpha,$$

whose expected value is

$$\boxed{\frac{180°}{11}}.$$

3/3/36. Let a, b be positive integers such that $a^2 \geq b$. Let

$$x = \sqrt{a+\sqrt{b}} - \sqrt{a-\sqrt{b}}.$$

(a) Prove that for all integers $a \geq 2$, there exists a positive integer b such that x is also a positive integer.

(b) Prove that for all sufficiently large a, there are at least two b such that x is a positive integer.

Note: We've received some questions about what is meant by "for all sufficiently large a." To give a simple example of this phrasing, it is true that for all sufficiently large positive integers n, we have $n^2 \geq 100$. Specifically, this is true for all $n \geq 10$.

Solution

(a) Let $a \geq 2$ and assume x is a positive integer. The latter is true if and only if x^2 is a perfect square. We have

$$x^2 = \left(\sqrt{a+\sqrt{b}}\right)^2 + \left(\sqrt{a-\sqrt{b}}\right)^2 - 2\left(\sqrt{a+\sqrt{b}}\right)\left(\sqrt{a-\sqrt{b}}\right),$$

which simplifies to

$$x^2 = 2a - 2\sqrt{a^2-b}.$$

Certainly x^2 is an integer, so $a^2 - b$ is a perfect square. That is, there exists a nonnegative integer k such that $k^2 = a^2 - b$. Then we can rewrite

$$x^2 = 2(a-k).$$

Moreover, x^2 is an *even* perfect square. That is, there exists some positive integer ℓ such that $x^2 = (2\ell)^2 = 4\ell^2$. Then $a - k = 2\ell^2$, so that $k = a - 2\ell^2$ and $a + k = 2a - 2\ell^2$.

The existence of k and ℓ therefore guarantees the existence of

$$b = a^2 - k^2 = 2\ell^2\left(2a - 2\ell^2\right) = 4\ell^2(a - \ell^2).$$

This is an integer, and if we let $\ell = 1$, then b is indeed positive if $a \geq 2$.

To check, let $\ell = 1$. Then $b = 4(a-1)$, and so

$$x^2 = 2a - 2\sqrt{a^2-4a+4} = 2a - 2\sqrt{(a-2)^2} = 2a - 2(a-2) = 4.$$

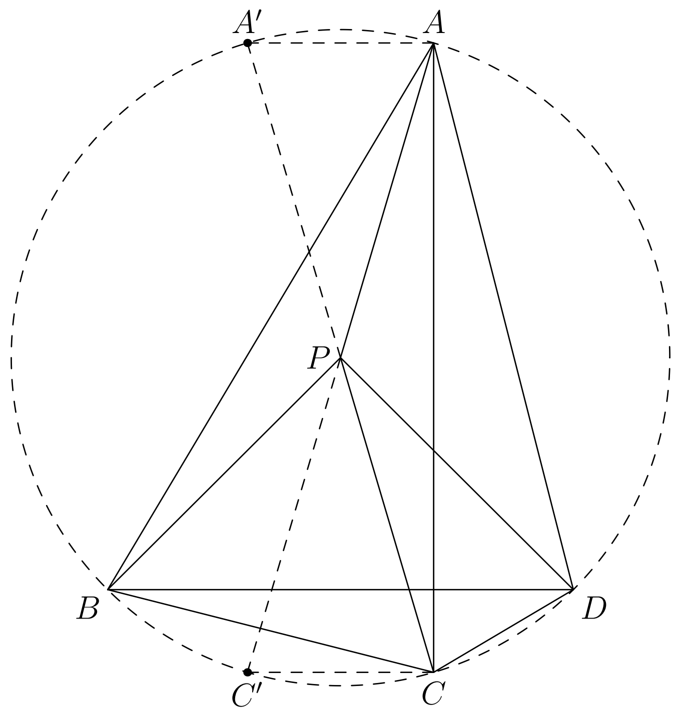
(b) We proved in the preceding part that the example $b = 4(a-1)$, i.e., setting $\ell = 1$, will work for all $a \geq 2$. Now consider $\ell = 2$, which gives $b = 16(a-4)$ and

$$x^2 = 2a - 2\sqrt{a^2 - 16a + 64} = 2a - 2(a-8) = 16.$$

Thus $x = 4$, independently of the value of a, for $a \geq 8$.

4/3/36. $ABCD$ is a convex quadrilateral where $\angle A = 45^\circ$ and $\angle C = 135^\circ$. P is a point strictly inside $\triangle ABC$ such that $\angle BAP = \angle CAD$ and $\angle BCP = \angle ACD$. Prove that $\overline{PB} \perp \overline{PD}$ if and only if $\overline{AC} \perp \overline{BD}$.

Solution



Let ω be the circumcircle of $ABCD$. Let C' be the other intersection of ω with line AP, and let A' be the other intersection of ω with line CP. Connect AA' and CC'.

Since $\angle BAP = \angle CAD$, we have $\overset{\frown}{BC'} = \overset{\frown}{CD}$. Similarly, since $\angle BCP = \angle ACD$, we have $\overset{\frown}{BA'} = \overset{\frown}{AD}$. This means that our three segments are parallel: $A'A \parallel BD \parallel C'C$. Thus by symmetry since P is the intersection of $A'C$ and AC', we have $PB = PD$.

Thus, $PB \perp PD$ if and only if P is the center of ω. This happens if and only if $A'C$ and AC' are diameters of ω, which is equivalent to $AC \perp A'A$ and $AC \perp C'C$, and since we showed above these are parallel to BD, this happens if and only if $AC \perp BD$ as desired.

5/3/36. Find all ordered triples of nonnegative integers (a, b, c) satisfying $2^a \cdot 5^b - 3^c = 1$.

Solution

The only ordered triples satisfying the given equation are $(a, b, c) = (1, 0, 0),\ (2, 0, 1),\ (1, 1, 2)$. These clearly work.

If $a = 0$, then 5^b and 3^c are both odd, so the LHS is even while the RHS is odd, contradiction. Hence $a > 0$.

If $c = 0$, then $2^a \cdot 5^b = 2$, giving the solution $(a, b, c) = (1, 0, 0)$. Henceforth assume $c > 0$.

If $a \geq 3$, taking the equation modulo 8 gives us $-3^c \equiv 1 \pmod 8$. The quantity 3^c can only be congruent to 1 or $3 \pmod 8$, so we get a contradiction. Hence $a = 1$ or $a = 2$.

Suppose $a = 2$, so that our equation reads $4 \cdot 5^b - 3^c = 1$. If $b = 0$, then we must have $c = 1$, giving us the solution $(a, b, c) = (2, 0, 1)$. Henceforth assume that $b > 0$. Taking the equation modulo 4 gives us $-3^c \equiv 1 \pmod 4$, so c must be odd. But taking the equation modulo 5 gives $-3^c \equiv 1 \pmod 5$, so $3^c \equiv 4 \pmod 5$ and hence $c \equiv 2 \pmod 4$. This implies c is even, a contradiction, so there are no other solutions in this case.

Hence suppose $a = 1$, so that our equation reads $2 \cdot 5^b - 3^c = 1$. Taking $b = 0$ gives us $(a, b, c) = (1, 0, 0)$ again, so suppose $b > 0$. Then taking our equation modulo 3 gives us $5^{b+1} \equiv 1 \pmod 3$, so $b + 1$ must be even and hence b must be odd. Therefore, let $b = 2d + 1$. Taking our equation modulo 5 then gives us $-3^c \equiv 1 \pmod 5$, so $c \equiv 2 \pmod 4$. Therefore, let $c = 4e + 2$. Now our equation reads

$$2 \cdot 5^{2d+1} - 3^{4e+2} = 1,$$

or $10 \cdot 5^{2d} - 9 \cdot 3^{4e} = 1$. Taking the equation modulo 9 gives us $6 \mid 2d$, so $3 \mid d$; hence let $d = 3f$, so our equation reads

$$10 \cdot 5^{6f} - 9 \cdot 3^{4e} = 1.$$

Taking the equation modulo 7 gives us $6 \mid 2e$, so $3 \mid e$; hence let $e = 3g$, so our equation reads

$$10 \cdot 5^{6f} - 9 \cdot 3^{12g} = 1.$$

Taking the equation modulo 13 gives us $12 \mid 6f$, so $2 \mid f$; hence let $f = 2h$, so our equation reads

$$10 \cdot 5^{12h} - 9 \cdot 3^{12g} = 1.$$

Now we prove a claim:

Claim: For all positive integers k, we have $5^{3\cdot 2^k} \equiv 3^{3\cdot 2^k} \equiv 2^{k+2}+1 \pmod{2^{k+3}}$.

Proof: By induction. The base case $k=1$ is trivial. Now suppose the claim is true for $k-1 \geq 1$, so that

$$5^{3\cdot 2^{k-1}} \equiv 2^{k+1}+1 \pmod{2^{k+2}}.$$

This means that for some constant C, we have

$$5^{3\cdot 2^{k-1}} = C \cdot 2^{k+2} + 2^{k+1} + 1.$$

Squaring both sides gives us

$$5^{3\cdot 2^k} = (C^2+C)\cdot 2^{2k+4} + C \cdot 2^{k+3} + 2^{2k+2} + 2^{k+2} + 1.$$

Since $k > 1$ we know $2k+4 > 2k+2 > k+3$, so reducing modulo 2^{k+3} gives us

$$5^{3\cdot 2^k} \equiv 2^{k+2}+1 \pmod{2^{k+3}}.$$

The inductive step for $3^{3\cdot 2^k}$ is similar. ■

Now, our claim tells us that taking our equation $10 \cdot 5^{12h} - 9 \cdot 3^{12g} = 1$ modulo 32, we have $10 \cdot 17^h - 9 \cdot 17^g \equiv 1 \pmod{32}$. Since $17^2 \equiv 1 \pmod{32}$ we must have $2 \mid h$ and $2 \mid g$, so we can rewrite our equation as $10 \cdot 5^{24h'} - 9 \cdot 3^{24g'} = 1$. Taking this modulo 64, we similarly find that $2 \mid h'$ and $2 \mid g'$, so we can rewrite this as $10 \cdot 5^{48h''} - 9 \cdot 3^{48g''} = 1$. Repeating this ad nauseum (using the fact that $(2^k+1)^2 \equiv 1 \pmod{2^{k+1}}$), we see that h and g, hence d and e must both be divisible by 2^k for all k, so we must have $d = e = 0$, giving us the solution $(a,b,c) = (1,1,2)$.

Hence the only solutions to our original equation are

$$(a,b,c) = (1,0,0),\ (2,0,1),\ (1,1,2).$$