USA Mathematical Talent Search

Round 2 Solutions

Year 36 — Academic Year 2024–2025

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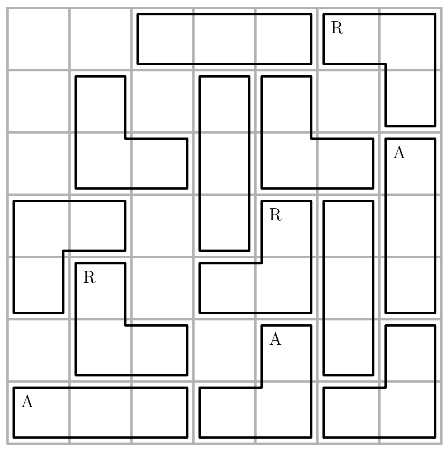
1/2/36. Fill each cell with an integer from 1-7 so each number appears exactly once in each row and column. In each "cage" of three cells, the three numbers must be valid lengths for the sides of a non-degenerate triangle. Additionally, if a cage has an "A", the triangle must be acute, and if the cage has an "R", the triangle must be right.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

3	1	7	6	2	R 4	5
4	7	2	5	6	1	3
7	3	5	2	1	6	A 4
1	2	6	4	R 5	3	7
2	R 5	1	3	4	7	6
6	4	3	1	A 7	5	2
A 5	6	4	7	3	2	1

2/2/36. In how many ways can a 3×3 grid be filled with integers from 1 to 12 such that all three of the following conditions are satisfied: (a) both 1 and 2 appear in the grid, (b) the grid contains at most 8 distinct values, and (c) the sums of the numbers in each row, each column, and both main diagonals are all the same? Rotations and reflections are considered the same.

Solution

The sum of all nine entries is equal to three times the magic sum. The sum of the central row and column and both diagonals is equal to four times the magic sum; it includes the center number four times and all other numbers once. Therefore, the central number is m where $3m$ is the magic sum. Let the top left and top right entries be $m + p$ and $m + q$ respectively; the rest of the magic square is now determined:

$m+p$	$m-p-q$	$m+q$
$m-p+q$	m	$m+p-q$
$m-q$	$m+p+q$	$m-p$

Reflecting in the downward diagonal interchanges q and $-q$, and reflecting in the upward diagonal interchanges p and $-p$, so we can assume that p and q are non-negative. Reflecting horizontally interchanges p and q, so we can assume $p \geq q$. The smallest number is $m-p-q$, which must be 1. The next smallest is $m - p$, which must be either 1 or 2.

If $m-p = 1$, then $q = 0$, so there are only three distinct numbers: $m-p$, m, and $m+p$. For these numbers to include 2, we must have $m = 2$ and $p = 1$; this gives one solution.

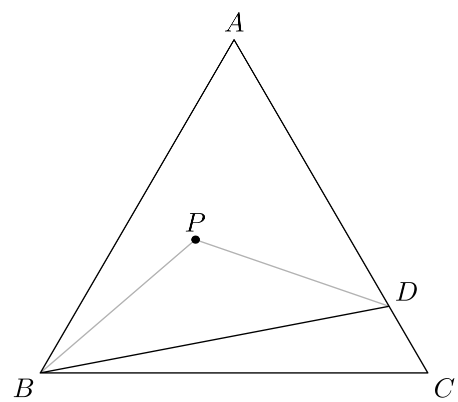
If $m-p = 2$, then $q = 1$, and the numbers are $m-p-1, m-p, m-p+1$; $m-1, m, m+1$; $m+p-1, m+p, m+p+1$. These numbers are all distinct if $p \geq 3$, so we can have $p = 1, m = 3$ giving five distinct numbers, or $p = 2, m = 4$ giving seven distinct numbers.

There are a total of $\boxed{3}$ solutions.

3/2/36. $\triangle ABC$ is an equilateral triangle. D is a point on $\overline{AC}$, and E is a point on $\overline{BD}$. Let P and Q be the circumcenters of $\triangle ABD$ and $\triangle AED$, respectively. Prove that $\triangle EPQ$ is an equilateral triangle if and only if $\overline{AB} \perp \overline{CE}$.

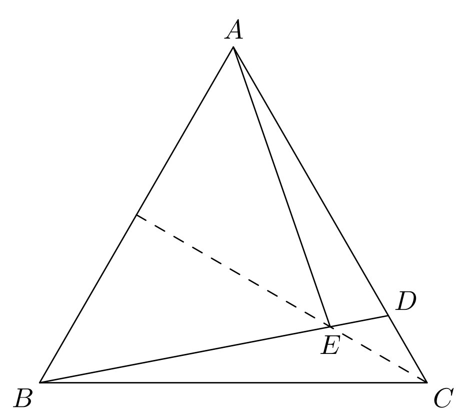
Solution

Since P is the circumcenter of triangle ABD, we have that $\angle BPD = 2\angle BAD = 120°$. Also, triangle BPD is isosceles with $BP = PD$, so the other two angles in triangle BPD are $\angle PBD = \angle PDB = 30°$.



We will show that $\overline{AB} \perp \overline{CE}$ if and only if triangle EPQ is equilateral through a series of equivalent to statements.

The condition $\overline{AB} \perp \overline{CE}$ holds if and only if E lies on the perpendicular bisector of $\overline{AB}$, which in turn is equivalent to $AE = BE$. Also, $AE = BE$ if and only if $\angle BAE = \angle ABE$.



Since $\angle AED$ is external to triangle ABE, we always have that

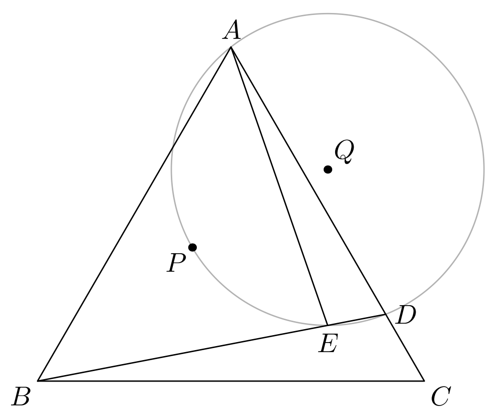
$$\angle BAE + \angle ABE = \angle AED.$$

Hence, $\angle BAE = \angle ABE$ if and only if $\angle AED = 2\angle ABD$.

Since P is the circumcenter of triangle ABD, we have that

$$\angle APD = 2\angle ABD.$$

Thus, $\angle AED = 2\angle ABD$ is equivalent to $\angle AED = \angle APD$. But this holds if and only if P lies on the circumcircle of triangle ADE.



If P lies on the circumcircle of triangle ADE, then $PQ = EQ$. Furthermore, we know that $\angle PDE = 30^\circ$, so $\angle PQE = 2\angle PDE = 60^\circ$. Therefore, triangle EPQ is equilateral.

Conversely, suppose triangle EPQ is equilateral. Then $PQ = EQ$, so P lies on the circumcircle of triangle ADE.

We conclude that$\overline{AB} \perp \overline{CE}$ if and only if triangle EPQ is equilateral.

4/2/36. Let $x_1 < x_2 < \cdots < x_n$ (with $n \geq 2$) and let S be the set of all the x_i. Let T be a randomly chosen subset of S. What is the expected value of the indexed alternating sum of T? Express your answer in terms of the x_i.

Note: We define the indexed alternating sum of T as

$$\sum_{i=1}^{|T|}(-1)^{i+1}(i)T[i],$$

where $T[i]$ is the ith element of T when listed in increasing order. For example, if $T = \{1, 3, 5\}$ then the indexed alternating sum of T is

$$1 \cdot 1 - 2 \cdot 3 + 3 \cdot 5 = 10.$$

Alternating sums of empty sets are defined to be 0.

Solution

Let $|S| = n$ and let x_i be the ith smallest element of S. We will compute the expected contribution of x_i to the sum.

Each x_i appears in T with probability $\frac{1}{2}$. If x_1 appears in T, it always has multiplicity 1. So, its expected contribution to the sum is $\frac{x_1}{2}$.

Similarly, x_2 appears in T with multiplicity 1 with probability $\frac{1}{4}$ (if x_1 is not in T) and multiplicity -2 with probability $\frac{1}{4}$ (if x_1 is in T). So x_2's expected contribution to the sum is $\frac{-x_2}{4}$.

Generalizing this, for $i \geq 3$, we have that x_i appears with multiplicity $(-1)^k(k+1)$ and corresponding probability $\frac{\binom{i-1}{k}}{2^i}$ for each $0 \leq k \leq i-1$. So, for $i \geq 3$, each x_i contributes an expected value of

$$\frac{x_i}{2^i}\sum_{k=0}^{i-1}(-1)^k(k+1)\binom{i-1}{k}.$$

This sum is equal to

$$\frac{x_i}{2^i}\left(\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k} + \sum_{k=0}^{i-1}(-1)^k k\binom{i-1}{k}\right).$$

The first summation is 0 by the binomial theorem. Then, for the second summation, note that $k\binom{i-1}{k} = (i-1)\binom{i-2}{k-1}$ by a committee-forming argument. Specifically, both sides tell us the number of ways we can form a committee of k members from an initial pool of $i-1$ people, with one member of the committee being the chair. For the left side, there are $\binom{i-1}{k}$

ways of choosing the committee members, and then k ways to choose which of the chosen members is the chair. For the right side, there are $i-1$ ways to choose the chair of the committee. Then there are $\binom{i-2}{k-1}$ ways to choose the remaining non-chair members.

So the second summation is equal to

$$(i-1)\sum_{k=1}^{i-1}(-1)^k\binom{i-2}{k-1}.$$

This is again 0 by the binomial theorem.

Thus, for $i \geq 3$ the entire sum is 0. The total expected value is

$$\boxed{\frac{x_1}{2}-\frac{x_2}{4}}.$$

5/2/36. Prove that there is no polynomial $P(x)$ with integer coefficients such that

$$P(\sqrt[3]{5}+\sqrt[3]{25}) = 2\sqrt[3]{5}+3\sqrt[3]{25}.$$

Solution

Denote $a = \sqrt[3]{5}+\sqrt[3]{25}$. We can first find a polynomial with integer coefficients that has a as a root by cubing a:

$$a^3 = 5 + 3(\sqrt[3]{5})^2\sqrt[3]{25} + 3\sqrt[3]{5}(\sqrt[3]{25})^2 + 25 = 30 + 15(\sqrt[3]{5}+\sqrt[3]{25}) = 30 + 15a.$$

This implies that a is a root of $P_1(x) = x^3 - 15x - 30$.

Next we show that $p\sqrt[3]{5}+q\sqrt[3]{25}$ is irrational for every rational p and q such that they are not both equal to 0. It can be shown that both $\sqrt[3]{5}$ and $\sqrt[3]{25}$ are irrational by assuming the contrary and cubing. (The idea is similar to the classic proof that $\sqrt{2}$ is irrational.) If either p or q is zero a similar argument shows that the number $p\sqrt[3]{5}+q\sqrt[3]{25}$ is irrational.
If we assume that $p\sqrt[3]{5}+q\sqrt[3]{25} = r$ is rational for some non-zero rationals then we square the equation $p\sqrt[3]{5}+q\sqrt[3]{25} = r$ and we get

$$p^2\sqrt[3]{25} + 5q^2\sqrt[3]{5} + 10pq = r^2.$$

Substituting $\sqrt[3]{25}$ with $\dfrac{r - p\sqrt[3]{5}}{q}$ we get

$$p^2\left(\frac{r - p\sqrt[3]{5}}{q}\right) + 5q^2\sqrt[3]{5} + 10pq = r^2.$$

Solving for $\sqrt[3]{5}$ we get

$$\sqrt[3]{5} = \frac{r^2 - 10pq - \frac{p^2 r}{q}}{5q^2 - \frac{p^3}{q}}.$$

Note that $5q^2 - \frac{p^3}{q} \neq 0$ since assuming the contrary will lead to $\sqrt[3]{5} = \frac{p}{q}$, which we know is false since $\sqrt[3]{5}$ is irrational. Additionally, $\sqrt[3]{5}$ being irrational contradicts our assumption that $p\sqrt[3]{5}+q\sqrt[3]{25}$ is a rational number. (If this expression were rational, then the right side would be the quotient of two rational numbers, and thus would be rational.)

Assume that there is $P(x)$ with integer coefficients such that $P(a) = 2\sqrt[3]{5}+3\sqrt[3]{25}$. Using polynomial division we can write

$$P(x) = P_1(x)q(x) + r(x)$$

for some polynomials with integer coefficients $q(x)$ and $r(x)$ such that $\deg r < \deg P = 3$. Using the above equations, the fact that $P_1(a) = 0$, and our assumption we can also deduce

$$2\sqrt[3]{5} + 3\sqrt[3]{25} = P(a) = P_1(a)q(a) + r(a) = r(a).$$

Thus, there exists a polynomial $r(x)$ with integer coefficients of degree at most 2 such that

$$r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}.$$

Our goal is to show that the last statement is false and this can be done by considering cases on $\deg r$.

Case 1: $\deg r = 1$. Let $r(x) = Ax + B$. If $r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$ then

$$A(\sqrt[3]{5} + \sqrt[3]{25}) + B = 2\sqrt[3]{5} + 3\sqrt[3]{25},$$

which in turn implies

$$(A-2)\sqrt[3]{5} + (A-3)\sqrt[3]{25} = -B.$$

We showed that the left-hand side of the above equation cannot be a rational number unless the coefficients in front of $\sqrt[3]{5}$ and $\sqrt[3]{25}$ are 0. However, we cannot have both $A - 2$ and $A - 3$ equal to 0. Thus, we cannot have $r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$ for any integers A and B.

Case 2: $\deg r = 2$. Let $r(x) = Ax^2 + Bx + C$. If $r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$ then

$$A(\sqrt[3]{25} + 10 + 5\sqrt[3]{5}) + B(\sqrt[3]{5} + \sqrt[3]{25}) + C = 2\sqrt[3]{5} + 3\sqrt[3]{25},$$

which can be rewritten as

$$(5A + B - 2)\sqrt[3]{5} + (A + B - 3)\sqrt[3]{25} = -C - 10.$$

Since the right-hand side is an integer, using the claim we proved earlier we must have $5A + B - 2 = 0$ and $A + B - 3 = 0$. Subtracting these two equations we get $4A + 1 = 0$ which does not have an integer solution for A.

This shows that there is no quadratic polynomial $r(x)$ such that $r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$, which concludes the proof in this case.

We proved there there does not exist a polynomial $r(x)$ with integer coefficients of degree at most 2 such that $r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$, which concludes the proof based on our setup using polynomial division.