USA Mathematical Talent Search

Round 1 Solutions

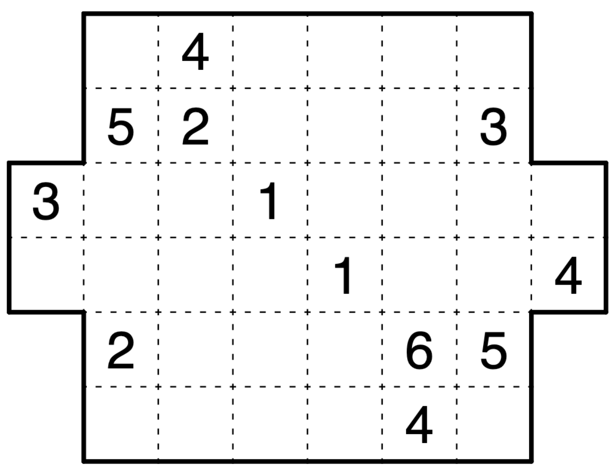
Year 36 — Academic Year 2024–2025

www.usamts.org

1/1/36. The "Manhattan distance" between two cells is the shortest distance between those cells when traveling up, down, left, or right, as if one were traveling along city blocks rather than as the crow flies.

Place numbers from 1-6 in some cells so the following criteria are satisfied:

1. A cell contains at most one number. Cells can be left empty.
2. For each cell containing a number N in the grid, exactly two other cells containing N are at a Manhattan distance of N.
3. For each cell containing a number N in the grid, no other cells containing N are at a Manhattan distance less than N.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

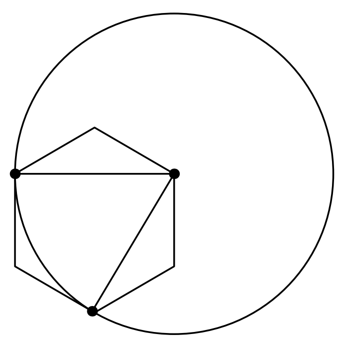
Solution

	3	4	6	3	5	4	
	5	2		2		3	
3	2		1	1	2		
6		3	1	1	3	2	4
	2	4	2		6	5	
		5		2	4	2	

2/1/36. A regular hexagon is placed on top of a unit circle such that one vertex coincides with the center of the circle, exactly two vertices lie on the circumference of the circle, and exactly one vertex lies outside of the circle. Determine the area of the hexagon.

Solution

The relationship between the regular hexagon and unit circle is shown below.



We use the side length of the hexagon, which we will call s, to compute its area. The top triangle is a $30-30-120$ triangle in which the two short sides are sides of the hexagon and the long side is the radius of the unit circle. Applying the Law of Cosines to this triangle gives us $1^2 = s^2 + s^2 - 2 \cdot s \cdot s \cdot \left(-\frac{1}{2}\right)$, which simplifies to $s = \frac{\sqrt{3}}{3}$.

A regular hexagon can be divided into 6 congruent equilateral triangles, each with side length s. The area of each equilateral triangle is $s^2 \cdot \frac{\sqrt{3}}{4}$, so the area of the hexagon is $6s^2 \cdot \frac{\sqrt{3}}{4} = s^2 \cdot \frac{3\sqrt{3}}{2}$. Substituting $s = \frac{\sqrt{3}}{3}$, the area of the hexagon is $\boxed{\frac{\sqrt{3}}{2}}$.

3/1/36. A sequence of integers $x_1, x_2, \ldots, x_k$ is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number.

The integers from 1 to 2024 are split into two groups, each written in increasing order. Group A is $a_1, a_2, \ldots, a_m$ and Group B is $b_1, b_2, \ldots, b_n$.

Find the largest integer M such that we can guarantee that we can pick M consecutive elements from either Group A or Group B which form a fibtastic sequence.

As an illustrative example, if a group of numbers is $2, 4, 11, 12, 13, 16, 18, 27, 29, 30$, the longest fibtastic sequence is $11, 12, 13, 16, 18$, which has length 5.

Note: We've received questions about what is meant by "Find the largest integer M such that we can guarantee ..." We mean "guarantee" in the sense that if we distribute 9 bananas to 2 monkeys, some monkey is guaranteed to get at least 5 bananas regardless of how the bananas are distributed, even though the other monkey will get fewer than 5 bananas.

Solution

We claim that the answer is $\boxed{M = 3}$. First, we show that we cannot do better than $M = 3$ by giving an example:

Group A: $1, 2, 3, \quad, \quad, \quad, 7, 8, 9, \quad, \quad, \quad, 13, 14, 15, \ldots$
Group B: $\quad, \quad, \quad, 4, 5, 6, \quad, \quad, \quad, 10, 11, 12, \quad, \quad, \quad, \ldots$

Here we see that for any four consecutive integers in a group, there is some pair with a difference of 4, which is not a Fibonacci number. So the maximum M in this case is 3.

Next, we show that $M = 3$ is always possible. To see this, consider the numbers 1, 2, 3, 4, 5. By the Pigeonhole Principle, some three of these must be in the same group. The consecutive differences among those three numbers must be at most 3 and this gives us a group of 3 fibtastic numbers. So $M = 3$ is always achievable.

4/1/36. During a lecture, each of 26 mathematicians falls asleep exactly once, and stays asleep for a nonzero amount of time. Each mathematician is awake at the moment the lecture starts, and the moment the lecture finishes. Prove that there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians such that there is some point in time during which all 6 are asleep.

Note: We consider a mathematician to be asleep at the moment they fall asleep, and awake at the moment they wake up.

Solution

One of the 26 mathematicians, who we will call Mathematician 26, must be the first to wake up from their nap. Since they are the first to wake up, at this moment all of the other mathematicians will either be asleep or will start their nap at this moment or later. Suppose 5 or more mathematicians are asleep at this instant. Since each mathematician sleeps for a nonzero amount of time, there must have been some previous point in time in which 6 mathematicians, including Mathematician 26, were all asleep, satisfying the problem. Thus, we consider the alternative in which 4 or fewer mathematicians are asleep when Mathematician 26 wakes up. In this scenario, at least 21 mathematicians start their nap at this time or later.

Before continuing, we consider the edge case in which the first two mathematicians to wake up do so at the same time. Suppose that two mathematicians, Mathematician 26 and Mathematician 25, are the first to wake up and that they wake up simultaneously. If 4 or more mathematicians are asleep at this time, we have a previous instant in which 6 mathematicians, including Mathematicians 26 and 25, were asleep as desired. If 3 or fewer mathematicians are asleep at the time Mathematicians 26 and 25 wake up, we still have at least 21 mathematicians who start their nap at this moment or later. An equivalent analysis applies if three or more mathematicians wake up simultaneously. Since multiple mathematicians waking up at the same time does not affect the number of mathematicians who fall asleep at that moment or later, we do not discuss this edge case further.

We return to the situation in which at least 21 mathematicians start their nap at the moment Mathematician 26 wakes up or later. One of these 21 mathematicians, who we will call Mathematician 21, is the first of these 21 mathematicians to wake up. If 5 or more mathematicians are asleep when Mathematician 21 wakes up, we have a previous instant in which 6 mathematicians were asleep at the same time. If not, we continue this process and perform the same analysis at the moments in which at least 16, 11, 6, and finally 1 mathematician start their nap at or after a particular time. If we never have 6 mathematicians who are asleep at the same time, then we have 6 mathematicians, specifically Mathematicians 1, 6, 11, 16, 21, and 26, whose sleep times don't overlap at all. Thus, there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians such that there is some point in time during which all 6 are asleep.

Solution

We begin by showing that a weaker-looking version of the problem is true:

> During a lecture, each of 26 mathematicians falls asleep exactly once, and stays asleep for a nonzero amount of time. Each mathematician is awake at the moment the lecture starts, and the moment the lecture finishes. Prove that there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians **such that for each pair of mathematicians, there is some point in time during which both mathematicians are asleep.**

Let M be the set of mathematicians. Build a digraph on M as follows: Draw a diedge from m_1 to m_2 if and only if the period of time m_1 is asleep falls entirely before the period of time m_2 is asleep (i.e., m_1 wakes up before m_2 falls asleep). Note that by construction, this digraph is transitive.

Now, decompose this graph as follows: Let V_0 be the set of vertices with indegree zero. For each vertex $m \in M \setminus V_0$, define $f(m)$ to be the maximum distance of m from any vertex in V_0, taken across all possible paths from vertices in V_0 to m. Then, for $k \geq 1$, let V_k be the set of vertices m such that $f(m) = k$. An equivalent construction is to let V_1 be the set of vertices whose in-neighborhood is in V_0, and V_k (for $k \geq 2$) be the set of vertices whose in-neighborhood is in $(V_0 \cup V_1 \cup \cdots \cup V_{k-1}) \setminus (V_0 \cup V_1 \cup \cdots \cup V_{k-2})$.

By the Pigeonhole Principle, since there are 26 vertices in our graph, we'll either get 6 sets in our decomposition, or some set will have 6 vertices. If we get 6 sets in our decomposition, we can find 6 vertices in $V_0,\ V_1,\ \ldots,\ V_5$ forming a directed path, giving us 6 mathematicians who fall asleep in sequence (so that no two are asleep at the same time). If we get 6 vertices in the same set, we know there are no diedges between any two vertices in that set (else one would be in a higher-numbered set), so the periods when the corresponding mathematicians are asleep must overlap.

Now we show that if there are n mathematicians such that for each pair, there is some point during which both are asleep, then there must be some point in time during which all n mathematicians are asleep. Represent mathematician i's sleeping time as an interval of real numbers $[a_i, b_i)$, and sort the intervals so that $a_1 \leq \cdots \leq a_n$. If there is some point during which mathematicians i and j are both asleep, we know that $a_j \leq b_i$ for all i and j. Hence for any i we know that $a_i \leq a_n \leq b_i$, so all mathematicians are asleep at time a_n. This strengthens the version of the problem at the start of our solution to the original problem.

Solution

A *partially ordered set* is a set with an operation $<$ that satisfies $x \not< x$, and $x < y$ and $y < z$ implies $x < z$. For example, the set of all positive integers, with $x < y$ whenever x is a proper divisor of y, is partially ordered. A *chain* in a partially ordered set is a subset which is totally ordered $x_1 < x_2 < \cdots < x_n$. An *antichain* is a subset in which no two elements are comparable; that is, no $x < y$.

Dilworth's Theorem: In a finite partial ordered set, if the largest antichain is of size n, the set can be partitioned into n chains. If the largest chain is of size n, the set can be partitioned into n antichains.

Make a partial order on the mathematicians in which mathematician X is less than mathematician Y if X wakes up at or before the time Y goes to sleep. Thus an antichain is a set of mathematicians for which every pair were asleep at the same time. Since each mathematician fell asleep only once, at the time the last mathematician in an antichain went to sleep, the others in that antichain must have all still been asleep, as anyone who had already woken up would have to go back to sleep in order to be asleep at the same time as the first one. Thus all mathematicians in an antichain were asleep at the same time. A chain is a set of mathematicians for which each one woke up at or before the time the next one went to sleep, so no two mathematicians in a chain were asleep at the same time.

If no six mathematicians were asleep at the same time, the largest antichain is of size at most 5. By Dilworth's Theorem, the set can be partitioned into 5 chains, and since $5 \cdot 5 < 26$, some chain must be of length 6, so we have a set of six mathematicians with no two asleep at the same time.

Alternatively, if there were no six mathematicians for which no two were asleep at the same time, there is no chain of length 6, so by Dilworth's Theorem, the set can be partitioned into 5 antichains. Since $5 \cdot 5 < 26$, there must be an antichain of size 6, and we have shown that all six mathematicians in that antichain were asleep at the same time.

5/1/36. Let $f(x) = x^2 + bx + 1$ for some real number b. Across all possible values of b, find all possible values for the number of integers x that satisfy $f(f(x) + x) < 0$.

That is, if there are some values of b that give us 180 integer solutions for x and there are other values of b that give us 314 integer solutions for x (and these are the only possibilities), the answer would be $\boxed{180, 314}$.

Solution

When $|b| \leq 2$, $f(x) \geq 0$ for all real x, so there is no solution to $f(f(x) + x) < 0$.

When $|b| > 2$, let r_1 and r_2 be the two roots of $f(x) = 0$, and WLOG, let $r_1 < r_2$. We can write $f(x) = (x - r_1)(x - r_2)$. Then we have:

$$\begin{aligned} f(f(x) + x) &= (f(x) + x - r_1)(f(x) + x - r_2) \\ &= ((x - r_1)(x - r_2) + x - r_1)((x - r_1)(x - r_2) + x - r_2) \\ &= (x - r_1)(x - (r_2 - 1))(x - (r_1 - 1))(x - r_2). \end{aligned}$$

Therefore, $f(f(x) + x)$ has four roots: $r_1 - 1$, r_1, $r_2 - 1$, and r_2.

CASE 1. $r_2 - r_1 < 1$. The solution to $f(f(x) + x) < 0$ is $(r_1 - 1, r_2 - 1) \cup (r_1, r_2)$.

Since $r_1 r_2 = 1$, either $0 < r_1 < 1 < r_2$, or $r_1 < -1 < r_2 < 0$. Since $r_2 - r_1 < 1$, we must have either $0 < r_1 < 1 < r_2 < 2$, or $-2 < r_1 < -1 < r_2 < 0$ respectively. Therefore, there is a single integer in (r_1, r_2), which also means there is a single integer in $(r_1 - 1, r_2 - 1)$. Therefore, 2 integers satisfy $f(f(x) + x) < 0$.

CASE 2. $r_2 - r_1 \geq 1$. The solution to $f(f(x) + x) < 0$ is $(r_1 - 1, r_1) \cup (r_2 - 1, r_2)$.

CASE 2.1. Neither r_1 nor r_2 is an integer. Then there is one integer in each of $(r_1 - 1, r_1)$ and $(r_2 - 1, r_2)$, giving us 2 integer values of x.

CASE 2.2. One of r_1 and r_2 is an integer. Then there is one integer in one of $(r_1 - 1, r_1)$ and $(r_2 - 1, r_2)$, and no integer in the other. In this case, $|b| = |r_1 + r_2| = m + \dfrac{1}{m}$ for some integer $m > 1$.

Note that r_1 and r_2 cannot be both integers, since $r_1 r_2 = 1$ and $r_1 \neq r_2$.

To sum up, the number of integers x that satisfy $f(f(x)+x) < 0$ is:

$$\begin{cases} 0, & \text{if } |b| \leq 2, \\ 1, & \text{if } |b| = m + \dfrac{1}{m} \text{ for some integer } m > 1, \\ 2, & \text{otherwise.} \end{cases}$$