USA Mathematical Talent Search

Round 3 Solutions

Year 35 — Academic Year 2023–2024

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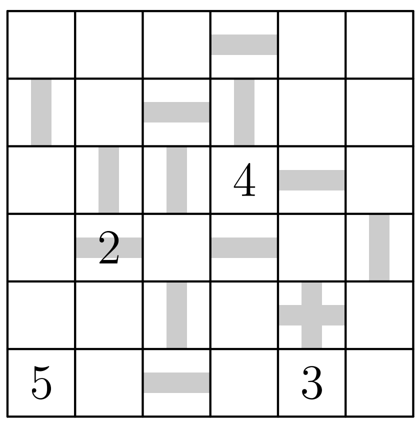
1/3/35. Fill in the grid with the numbers 1 to 6 so that each number appears exactly once in each row and column. A horizontal gray line marks any cell when it is the middle cell of the three consecutive cells with the largest sum in that row. Similarly, a vertical gray line marks any cell when it is the middle of the three consecutive cells with the largest sum in that column. If there is a tie, multiple lines are drawn in the row or column. A cell can have both lines drawn, with the appearance of a plus sign.

For example, if a filled row had 1 5 4 2 3 6, horizontal lines would be drawn in the two cells with 4 and 3, since out of $1+5+4$, $5+4+2$, $4+2+3$, and $2+3+6$, the largest sums are $5+4+2$ and $2+3+6$.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

6	1	3	5	4	2
2	6	5	3	1	4
3	5	1	4	2	6
4	2	6	1	5	3
1	3	4	2	6	5
5	4	2	6	3	1

2/3/35. Grogg takes an $a \times b \times c$ rectangular block (where a, b, c are positive integers), paints the outside of it purple, and cuts it into abc small $1 \times 1 \times 1$ cubes. He then places all the small cubes into a bag, and Winnie reaches in and randomly picks one of the small cubes. If the probability that Winnie picks a totally unpainted cube is 20%, determine all possible values of the number of cubes in the bag.

Solution

We find bounds on a, b, and c.

We have the equation

$$\frac{20}{100} = \frac{1}{5} = \frac{(a-2)(b-2)(c-2)}{abc} = \frac{a-2}{a} \cdot \frac{b-2}{b} \cdot \frac{c-2}{c}.$$

Suppose WLOG $a \le b \le c$. Then $\frac{a-2}{a} \le \frac{b-2}{b} \le \frac{c-2}{c}$, and in particular we have

$$\frac{1}{5} \le \frac{a-2}{a} \le \sqrt[3]{\frac{1}{5}}.$$

This simplifies to

$$\frac{1}{5} \le 1 - \frac{2}{a} \le \frac{1}{\sqrt[3]{5}},$$

or

$$\frac{4}{5} \ge \frac{2}{a} \ge 1 - \frac{1}{\sqrt[3]{5}},$$

or

$$\frac{5}{2} = 2.5 \le a \le \frac{2}{1 - \frac{1}{\sqrt[3]{5}}} \approx 4.817.$$

This bounds a to be 3 or 4.

If $a = 3$ then we have $\frac{3}{5} = \frac{b-2}{b} \cdot \frac{c-2}{c}$, and in particular

$$\frac{3}{5} \le 1 - \frac{2}{b} \le \frac{\sqrt{3}}{\sqrt{5}}.$$

So

$$\frac{2}{5} \ge \frac{2}{b} \ge 1 - \frac{\sqrt{3}}{\sqrt{5}},$$

or

$$5 \le b \le \frac{2}{1 - \frac{\sqrt{3}}{\sqrt{5}}} \approx 8.873.$$

So we have the bounds $5 \le b \le 8$. Solving for $c = 5 \cdot \frac{b-2}{b-5}$ for $5 \le b \le 8$ only yields positive integers for $(a,b,c) = (3,6,20)$ and $(a,b,c) = (3,8,10)$.

If $a = 4$ then we have $\dfrac{2}{5} = \dfrac{b-2}{b} \cdot \dfrac{c-2}{c}$, and in particular

$$\frac{2}{5} \le 1 - \frac{2}{b} \le \frac{\sqrt{2}}{\sqrt{5}}.$$

So

$$\frac{3}{5} \ge \frac{2}{b} \ge 1 - \frac{\sqrt{2}}{\sqrt{5}},$$

or

$$3 \le b \le \frac{2}{1 - \frac{\sqrt{2}}{\sqrt{5}}} \approx 5.44.$$

So (recalling $4 = a \le b$) we have the bounds $4 \le b \le 5$. Solving for $c = 10 \cdot \frac{b-2}{3b-10}$ yields positive integers for $(a,b,c) = (4,4,10)$ and $(a,b,c) = (4,5,6)$.

So the possible sides of the boxes are $(3,6,20)$, $(3,8,10)$, $(4,4,10)$, and $(4,5,6)$. The number of unit cubes in the bag is equivalent to the volume of the box; these boxes have volumes 360, 240, 160, and 120, respectively.

3/3/35. Lizzie and Alex are playing a game on the whiteboard. Initially, n twos are written on the board. On a player's turn they must either

1. change any single positive number to 0, or
2. subtract one from any positive number of positive numbers on the board.

The game ends once all numbers are 0, and the last player who made a move wins. If Lizzie always plays first, find all n for which Lizzie has a winning strategy.

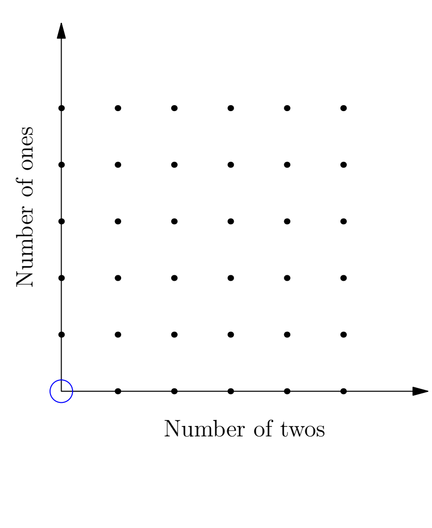
Solution

Any legal position in the game can be described by a pair of nonnegative integers (x, y) with x being the number of twos on the board and y being the number of ones on the board. Call a position "hot" if it is a position where the player whose turn it is to move will win with best play, and call a position "cold" if the player whose turn it is to move will lose with best play. The winning strategy from a hot position is to make a move that leaves your opponent with a cold position.

To classify positions into hot and cold, we can use a recursive approach using the following three rules:

1. $(0, 0)$ is a cold position. (If it's your turn and the board is empty, then the other player won).
2. Any position that can reach a cold position in a single move is **hot**.
3. Any position from which all moves lead to a hot position is **cold**.

We can visualize positions in the xy-plane so that the moves have a geometrical interpretation. Before a position has been classified, we draw it with a solid black dot. We use an open blue circle for a cold position and a solid red circle for a hot position. We can start by coloring the origin cold, as below:

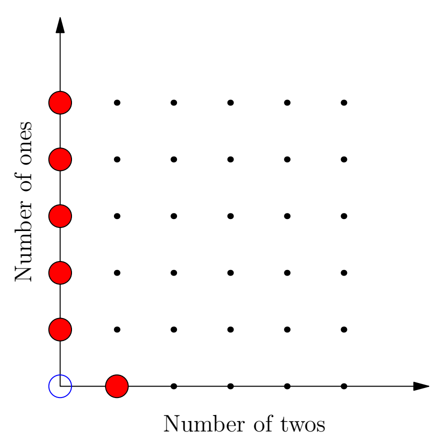


Then any position from which we can reach the origin (which is cold) in one move is hot. We can re-describe our two legal move options geometrically to interpret them in our picture:

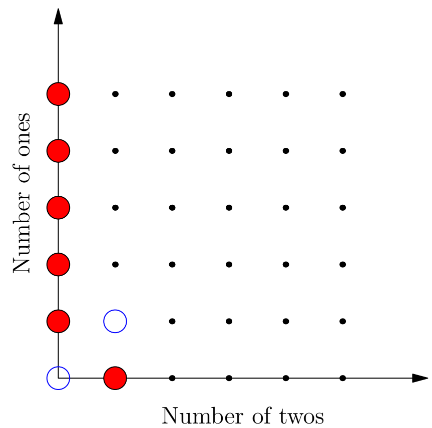
1. Changing a single positive number to 0 means from any point in the grid we can take a single step either south or west.

2. Subtracting one from any positive number of positive numbers on the board means we can either take any number of steps south (in the case where we subtracting one from ones only), or we can move northwest any number of steps (in the case where we are subtracting one from twos only, effectively exchanging twos for ones), or we can move along some combination of first south, then northwest (removing ones and exchanging twos for ones simultaneously). Note that the order matters here: we cannot go northwest and then south, since that would require two moves (first, exchanging twos for ones, and second, removing those newly acquired ones)

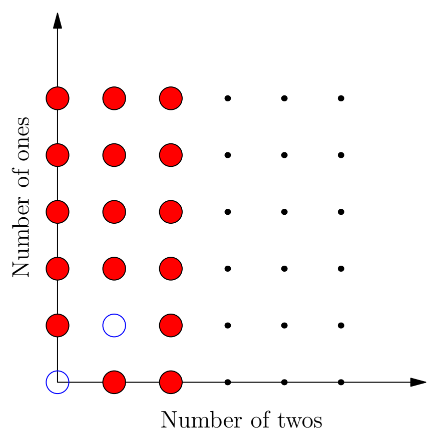
From these interpretations, we can see that every point of the form $(0, y)$ with $y > 0$ is hot, and the point $(1, 0)$ is also hot:



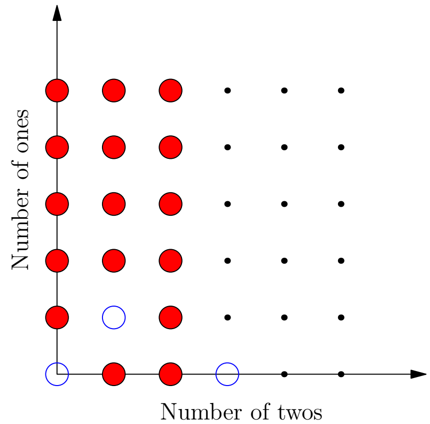
Then the point $(1, 1)$ is cold, since everywhere it reaches in one move is hot:



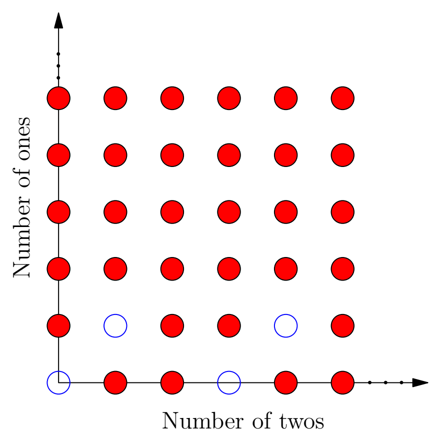
The locus of points that can reach the cold point $(1,1)$ is then all the remaining uncategorized points on the line $x=1$ as well as all the points on the line $x=2$.



Then, the point $(3,0)$ will be cold, since it can only reach hot positions in one move:



At this point, due to the geometry of the moves, the entire pattern will continue to repeat in cycles of 3 columns. So the grid will be colored as follows:



We see that when $y = 0$, the x for which Lizzie has a winning strategy for are $x = 1, 2, 4, 5, \ldots$, that is, the values of x that are of the form $\boxed{3k + 1 \text{ or } 3k + 2}$. The overall winning strategy is to always make moves that leave Alex with a position of the form $(3j, 0)$ or $(3j + 1, 1)$.

4/3/35. In this problem, a *simple polygon* is a polygon that does not intersect itself and has no holes, and a *side* of a polygon is a maximal set of collinear, consecutive line segments in the polygon. In particular, we allow two or more consecutive vertices in a simple polygon to be identical, and three or more consecutive vertices in a simple polygon to be collinear. By convention, polygons must have at least three sides. A simple polygon is *convex* if every one of its interior angles is 180° or less. A simple polygon is *concave* if it is not convex.

Let P be the plane. Prove or disprove each of the following statements:

(a) There exists a function $f : P \to P$ such that for all positive integers $n \geq 4$, if $v_1, v_2, \ldots, v_n$ are the vertices of a simple concave n-sided polygon in some order, then $f(v_1), f(v_2), \ldots, f(v_n)$ are the vertices of a simple convex polygon in some order (which may or may not have n sides).

(b) There exists a function $f : P \to P$ such that for all positive integers $n \geq 4$, if $v_1, v_2, \ldots, v_n$ are the vertices of a simple convex n-sided polygon in some order, then $f(v_1), f(v_2), \ldots, f(v_n)$ are the vertices of a simple concave polygon in some order (which may or may not have n sides).

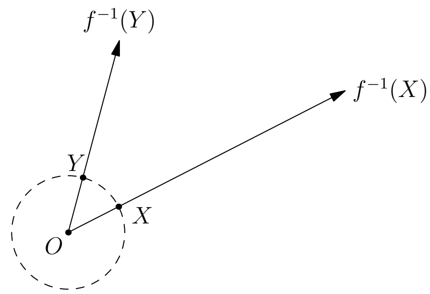
Solution

(a) Such a function does exist. Let O be the origin, and define $f : P \to P$ by:

$$f(X) = \begin{cases} \text{The intersection point of } \overrightarrow{OX} \text{ and the unit circle} & \text{if } X \neq O, \\ (1,0) & \text{otherwise.} \end{cases}$$

Since the image of f is the unit circle, any 3 or more distinct points in the image will form the vertices of a convex polygon. Thus, we only need to show that if $v_1, v_2, \ldots v_n$ are the vertices of a concave polygon, then the set $\{f(v_1), f(v_2), \ldots, f(v_n)\}$ contains at least 3 distinct points. To do this, we can show the contrapositive: If $\{f(v_1), f(v_2), \ldots, f(v_n)\} = \{X, Y\}$ for not-necessarily-distinct points X and Y, then $v_1, v_2, \ldots v_n$ do not form a concave polygon.

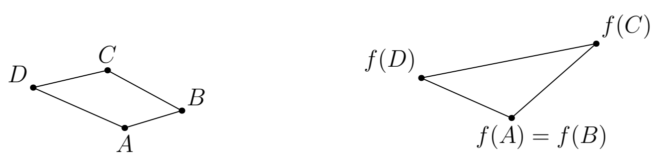
So, suppose $\{f(v_1), f(v_2), \ldots, f(v_n)\} = \{X, Y\}$. The inverse images of X and Y are the rays $\overrightarrow{OX}$ and $\overrightarrow{OY}$, possibly including the origin. Thus, $v_1, v_2, \ldots, v_n$ all lie on $\overrightarrow{OX} \cup \overrightarrow{OY}$.



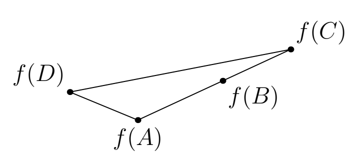
However, it is not possible to construct a concave polygon with vertices all on $\overrightarrow{OX} \cup \overrightarrow{OY}$. Thus, f has the desired property.

(b) Such a function does not exist. We will show this by contradiction. Assume $f : P \to P$ maps convex polygons with $n \geq 4$ vertices $v_1, v_2, \ldots v_n$ to concave polygons.

First, we notice that f must be injective. Otherwise, if $f(A) = f(B)$ for $A \neq B$, then we can pick any C and D such that $ABCD$ is convex, and there will be no way for $f(A)$, $f(B)$, $f(C)$, and $f(D)$ to form a concave polygon since the 4 points will always be collinear or form a triangle.

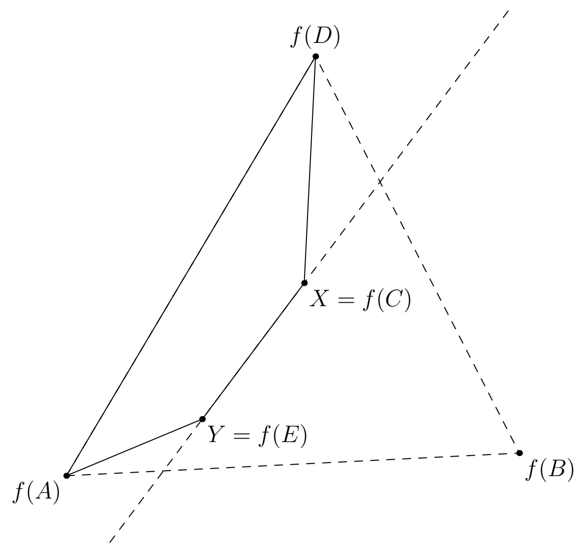


Next, notice that if A, B, and C are not collinear, then $f(A)$, $f(B)$, and $f(C)$ are not collinear as well. Otherwise, if $f(A)$, $f(B)$, and $f(C)$ were collinear, we can pick any point D such that $ABCD$ is convex, and there will be no way for $f(A)$, $f(B)$, $f(C)$, and $f(D)$ to form a concave polygon, like above.



Finally, let $ABCDE$ be a regular pentagon. Then, $f(A)$, $f(B)$, $f(C)$, $f(D)$, and $f(E)$ will be 5 distinct points, no 3 collinear. Suppose that the convex hull of these points is a quadrilateral or a pentagon. Then, we can take the inverse image of this convex hull to get a convex polygon, which violates the condition.

Otherwise, the convex hull is a triangle. In that case, let X and Y be the two points on the inside. Two points of the convex hull fall on the same side of $\overleftrightarrow{XY}$. Those two points together with X and Y form a convex quadrilateral whose inverse image violates the condition.



Thus, in either case, we have a contradiction, so such an f can't exist.

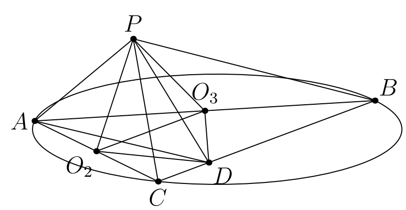
5/3/35. Let ω be the unit circle in the xy-plane in 3-dimensional space. Find all points P not on the xy-plane that satisfy the following condition: There exist points A, B, and C on ω such that

$$\angle APB = \angle APC = \angle BPC = 90^\circ.$$

Solution

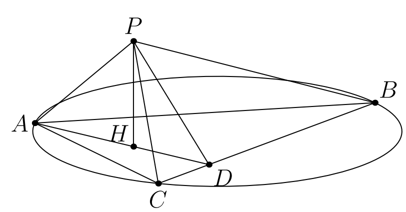
Let A, B, C, and P be points that satisfy the given conditions. Let D be the projection of A onto $\overline{BC}$. Let O_2 be the midpoint of $\overline{AC}$, and let O_3 be the midpoint of $\overline{AB}$. Then D is the reflection of A in $\overline{O_2O_3}$.

Since $\angle APB = 90^\circ$, P lies on the sphere centered at O_3 with radius $\frac{AB}{2}$. Also, $\angle ADB = 90^\circ$, so D also lies on this sphere. Likewise, P and D lie on the sphere centered at O_2 with radius $\frac{AC}{2}$.



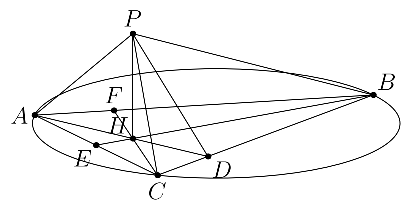
The intersection of these two spheres is a circle that contains A, P, and D. Also, the plane of this circle is perpendicular to $\overline{O_2O_3}$. Since $\overline{BC}$ is parallel to $\overline{O_2O_3}$, plane APD is also perpendicular to $\overline{BC}$. It follows that $\overline{PD}$ is perpendicular to $\overline{BC}$.

Let H be the projection of P onto plane ABC.



Then $\overline{PH}$ is perpendicular to $\overline{BC}$. Both $\overline{PD}$ and $\overline{PH}$ are perpendicular to $\overline{BC}$, so plane PHD is perpendicular to $\overline{BC}$. But $\overline{AD}$ is perpendicular to $\overline{BC}$, so plane PHD contains point A. Furthermore, points D and H lie in plane ABC, so H lies on altitude AD.

Similarly, H lies on altitudes BE and CF of triangle ABC, so H is the orthocenter of triangle ABC.



Let O be the center of circle ω, and let G be the centroid of triangle ABC. We can place the diagram in coordinate space, so that circle ω is the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $z = 0$. Let the coordinates of A, B, C be $(a_1, a_2, 0)$, $(b_1, b_2, 0)$, $(c_1, c_2, 0)$, respectively, so $a_1^2 + a_2^2 = b_1^2 + b_2^2 = c_1^2 + c_2^2 = 1$. Also, the coordinates of G are given by

$$G = \left(\frac{a_1 + b_1 + c_1}{3}, \frac{a_2 + b_2 + c_2}{3}, 0 \right).$$

From the Euler line, we know that O, G, and H are collinear with $OG : OH = 1 : 3$. So the coordinates for H are given by

$$H = (a_1 + b_1 + c_1, a_2 + b_2 + c_2, 0).$$

Then the coordinates of P are $(a_1 + b_1 + c_1, a_2 + b_2 + c_2, z)$ for some real number z.

Since $\angle APB = 90^\circ$, we have that $PA^2 + PB^2 = AB^2$. We can compute these quantities, as follows:

$$\begin{aligned} PA^2 &= AH^2 + HP^2 \\ &= (a_2 + a_3)^2 + (b_2 + b_3)^2 + z^2, \\ PB^2 &= BH^2 + HP^2 \\ &= (a_1 + a_3)^2 + (b_1 + b_3)^2 + z^2, \\ AB^2 &= (a_1 - a_2)^2 + (b_1 - b_2)^2. \end{aligned}$$

Thus,

$$\begin{aligned} &(a_2 + a_3)^2 + (b_2 + b_3)^2 + z^2 + (a_1 + a_3)^2 + (b_1 + b_3)^2 + z^2 \\ &= (a_1 - a_2)^2 + (b_1 - b_2)^2. \end{aligned}$$

Using the fact that $a_3^2 + b_3^2 = 1$, this simplifies to

$$2a_1a_2 + 2a_1a_3 + 2a_2a_3 + 2b_1b_2 + 2b_1b_3 + 2b_2b_3 + 2z^2 + 2 = 0.$$

Finally, letting $x = a_1 + a_2 + a_3$ and $y = b_1 + b_2 + b_3$, we find that

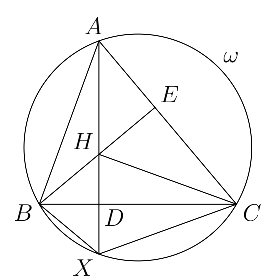
$$\begin{aligned} x^2 + y^2 + 2z^2 &= (a_1 + a_2 + a_3)^2 + (b_1 + b_2 + b_3)^2 + 2z^2 \\ &= (a_1^2 + a_2^2 + a_3^2 + 2a_1a_2 + 2a_1a_3 + 2a_2a_3) \\ &\quad + (b_1^2 + b_2^2 + b_3^2 + 2b_1b_2 + 2b_1b_3 + 2b_2b_3) \\ &= (a_1^2 + b_1^2) + (a_2^2 + b_2^2) + (a_3^2 + b_3^2) \\ &\quad + (2a_1a_2 + 2a_1a_3 + 2a_2a_3 + 2b_1b_2 + 2b_1b_3 + 2b_2b_3 + 2z^2) \\ &= 1 + 1 + 1 - 2 \\ &= 1. \end{aligned}$$

Thus, P lies on the curve $x^2 + y^2 + 2z^2 = 1$, which is an ellipsoid. It can be generated by taking the sphere $x^2 + y^2 + z^2 = 1$ and dilating it towards the plane of circle ω.

Next, we show that all points on the curve $x^2 + y^2 + 2z^2 = 1$ satisfy the given condition, except for the points where $z = 0$. (In other, the points on the circle ω are the exceptions.) If $z = 0$, then P is in plane ABC. But no point P in plane ABC can satisfy $\angle APB = \angle APC = \angle BPC = 90^\circ$.

Otherwise, let P be a point on the curve $x^2 + y^2 + 2z^2 = 1$, where $z \neq 0$. Let H be the projection of P onto plane ABC. First, we find points A, B, C on ω such that H is the orthocenter of triangle ABC. (Since $x^2 + y^2 < 1$, H lies in the interior of ω.)

Let line AH intersect ω again at X. Let D be the midpoint of $\overline{HX}$, and let the perpendicular bisector of $\overline{HX}$ intersect ω at B and C, and let line BH intersect $\overline{AC}$ at E. Triangle BXC is the reflection of triangle BHC in $\overline{BC}$, so $\angle HBC = \angle XBC$.



Quadrilateral $ACXB$ is cyclic, so $\angle CAX = \angle CBX$. Therefore, $\angle HBC = \angle CAX$. In other words, $\angle EBD = \angle EAD$, so quadrilateral $AEDB$ is cyclic. Hence, $\angle AEB = \angle ADB = 90^\circ$. This means H lies on altitudes $\overline{AD}$ and $\overline{BE}$ of triangle ABC, so H is the orthocenter of triangle ABC.

Let the coordinates of A, B, C be $(a_1, a_2, 0)$, $(b_1, b_2, 0)$, $(c_1, c_2, 0)$, respectively. Then the coordinates of H are $(x, y, 0) = (a_1 + b_1 + c_1, a_2 + b_2 + c_2, 0)$, where $x = a_1 + b_1 + c_1$ and

$y = a_2 + b_2 + c_2$. Then

$$\begin{aligned} x^2 + y^2 &= OH^2 \\ &= (a_1 + b_1 + c_1)^2 + (a_2 + b_2 + c_2)^2 \\ &= (a_1^2 + a_2^2) + (b_1^2 + b_2^2) + (c_1^2 + c_2^2) \\ &\quad + (2a_1b_1 + 2a_1c_1 + 2b_1c_1) + (2a_2b_2 + 2a_2c_2 + 2b_2c_2) \\ &= (2a_1b_1 + 2a_1c_1 + 2b_1c_1) + (2a_2b_2 + 2a_2c_2 + 2b_2c_2) + 3. \end{aligned}$$

Since x, y, z satisfy $x^2 + y^2 + 2z^2 = 1$, we have that

$$(2a_1b_1 + 2a_1c_1 + 2b_1c_1) + (2a_2b_2 + 2a_2c_2 + 2b_2c_2) + 2z^2 + 2 = 0.$$

(At this point, you may recognize that we are reversing the calculations earlier in our work.)

Then

$$\begin{aligned} PA^2 + PB^2 - AB^2 &= (AH^2 + HP^2) + (BH^2 + HP^2) - AB^2 \\ &= (a_2 + a_3)^2 + (b_2 + b_3)^2 + z^2 \\ &\quad + (a_1 + a_3)^2 + (b_1 + b_3)^2 + z^2 \\ &\quad - [(a_1 - a_2)^2 + (b_1 - b_2)^2] \\ &= (2a_1b_1 + 2a_1c_1 + 2b_1c_1) + (2a_2b_2 + 2a_2c_2 + 2b_2c_2) + 2z^2 + 2 \\ &= 0, \end{aligned}$$

so $PA^2 + PB^2 = AB^2$. By the Pythagorean Theorem, $\angle APB = 90^\circ$. Similarly, we can prove that $\angle APC = 90^\circ$ and $\angle BPC = 90^\circ$. Thus, P satisfies the given conditions.

The set of points P that we seek is then is the set of points that satisfy $x^2 + y^2 + 2z^2 = 1$, where $z \neq 0$.