USA Mathematical Talent Search
Round 2 Solutions
Year 35 — Academic Year 2023–2024
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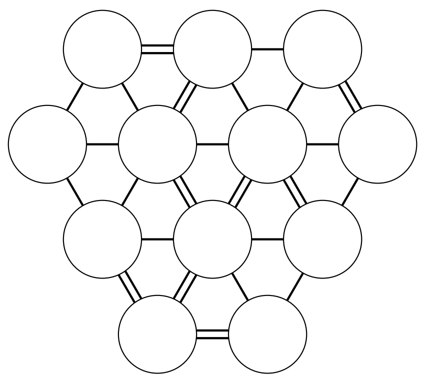
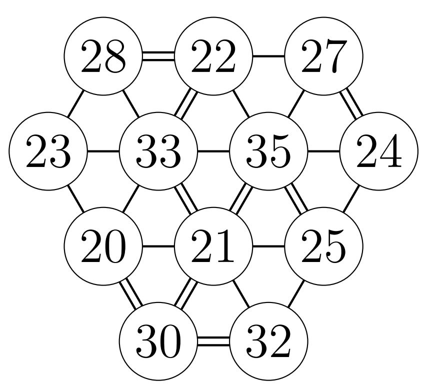
1/2/35. In the diagram below, fill the 12 circles with numbers from the following bank so that each number is used once. Two circles connected by a single line must contain relatively prime numbers. Two circles connected by a double line must contain numbers that are not relatively prime.

Bank: 20, 21, 22, 23, 24, 25, 27, 28, 30, 32, 33, 35



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/2/35. Malmer Pebane's apartment uses a six-digit access code, with leading zeros allowed. He noticed that his fingers leave smudges that reveal which digits were pressed. He decided to change his access code to provide the largest number of possible combinations for a burglar to try when the digits are known. For each number of distinct digits that could be used in the access code, calculate the number of possible combinations when the digits are known but their order and frequency are not known. For example, if there are smudges on 3 and 9, two possible codes are 393939 and 993999. Which number of distinct digits in the access code offers the most combinations?

Solution

We find the number of possible codes for each number of distinct digits.

1 digit: Since the digit is known, there is only 1 possible code. For example, if there is a smudge on 2, the only possible code is 222222.

2 digits: There are two choices for each of the positions in the six-digit code, giving us an initial count of $2^6 = 64$ codes. However, we need to subtract the two codes in which the same digit appears in all six positions. This gives us $64 - 2 = 62$ possible codes.

3 digits: We start with $3^6 = 729$ possible codes, but need to subtract the codes in which only one or two distinct digits are used. For the codes with two distinct digits, there are $\binom{3}{2}$ ways to choose which two digits are used, and our work for the preceding case tells us that there are 62 possible codes for each choice of two digits. So, there are $\binom{3}{2} \cdot 62 = 186$ codes containing two distinct digits. Similarly, there are $\binom{3}{1} \cdot 1 = 3$ codes containing 1 distinct digit. Thus, we have $729 - 186 - 3 = 540$ possible codes.

4 digits: Using similar reasoning, there are $4^6 - \binom{4}{3} \cdot 540 - \binom{4}{2} \cdot 62 - \binom{4}{1} \cdot 1 = 1560$ possible codes.

5 digits: There are $5^6 - \binom{5}{4} \cdot 1560 - \binom{5}{3} \cdot 540 - \binom{5}{2} \cdot 62 - \binom{5}{1} \cdot 1 = 1800$ possible codes.

6 digits: Each of the six distinct digits must be used exactly once, giving us $6! = 720$ possible codes.

The number of possible codes is maximized when there are $\boxed{5}$ distinct digits.

3/2/35. We say that three numbers are *balanced* if either all three numbers are the same, or they are all different. A grid consisting of hexagons is presented in Figure 1. Each hexagon is filled with the number 1, 2, or 3, so that for any three hexagons that are mutually adjacent and oriented with two hexagons on the bottom and one hexagon on the top (as in Figure 3), the three numbers in the hexagons are balanced. Prove that when the grid is filled completely, the three numbers in the three shaded hexagons are balanced.

(An example of a partially filled-in grid is shown in Figure 2. There are other ways of filling in the grid.)

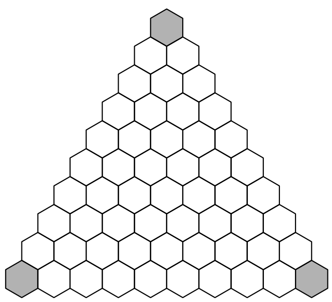


Figure 1

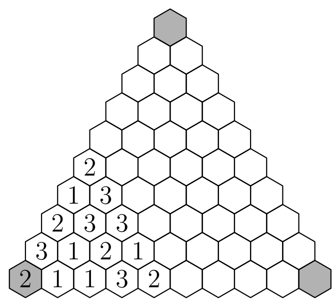


Figure 2

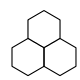
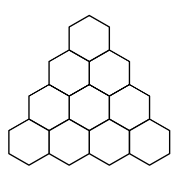


Figure 3

Solution

Let x, y, z be three numbers, where each is equal to 1, 2, or 3. Then the numbers x, y, z are balanced if and only if $x + y + z \equiv 0 \pmod 3$.

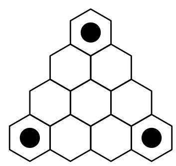
Consider the following portion of the grid.



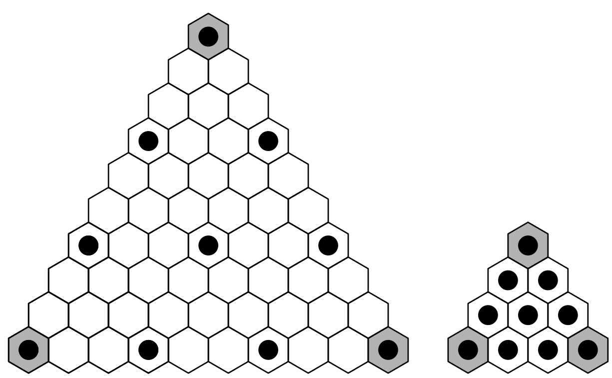
Let a, b, c, d be the numbers in the bottom row. Then we can write expressions for the other numbers, based on the modulo 3 condition. (For example, the number above a and b is equivalent to $-a - b$.)

$$-a-3b-3c-d$$
$$a+2b+c \qquad\qquad b+2c+d$$
$$-a-b \qquad\qquad -b-c \qquad\qquad -c-d$$
$$a \qquad\qquad b \qquad\qquad c \qquad\qquad d$$

Note that $a+d+(-a-3b-3c-d) = -3b-3c \equiv 0 \pmod 3$. Thus, the three numbers in the marked hexagons below are balanced.

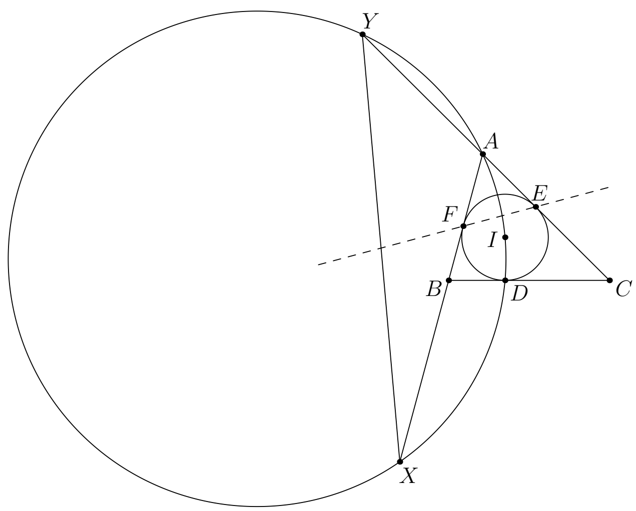


So if we extract the marked hexagons below from the original grid to form another grid, it too satisfies the condition in the problem. Applying the same result, we obtain that the three numbers in the three shaded hexagons are balanced.



4/2/35. The incircle of triangle ABC with $AB \neq AC$ has center I and is tangent to BC, CA, and AB at D, E, and F respectively. The circumcircle of triangle ADI intersects AB and AC again at X and Y. Prove that EF bisects XY.

Solution



We use directed angles to avoid configuration issues. Since

$$\measuredangle IXY = \measuredangle IAY = \measuredangle XAI = \measuredangle XYI,$$

$IX = IY$. So, $FX = \sqrt{IX^2 - FI^2} = \sqrt{IY^2 - EI^2} = EY$. Now,

$$\begin{aligned} \text{Pow}_{(AEF)}X - \text{Pow}_{(DEF)}X &= FX \cdot AX - FX^2 \\ &= FX \cdot AF \\ &= EY \cdot AE \\ &= EY^2 - EY \cdot AY \\ &= \text{Pow}_{(DEF)}(Y) - \text{Pow}_{(AEF)}(Y). \end{aligned}$$

Thus, the midpoint of $\overline{XY}$ lies on $\overline{EF}$, by linearity of power.

Solution

Since $\measuredangle IXY = \measuredangle IAY = \measuredangle XAI = \measuredangle XYI$, the midpoint of $\overline{XY}$ is the foot of the altitude from I to $\overline{XY}$; let's call this point M. Then I's Simson Line with respect to triangle AXY tells us that E, F, and M are collinear. This collinearity gives us the desired conclusion that $\overline{EF}$ bisects $\overline{XY}$.

5/2/35. Let m and n be positive integers. Let S be the set of all points (x, y) with integer coordinates such that $1 \le x, y \le m+n-1$ and $m+1 \le x+y \le 2m+n-1$. Let L be the set of the $3m+3n-3$ lines parallel to one of $x = 0$, $y = 0$, or $x+y = 0$ and passing through at least one point in S. For which pairs (m, n) does there exist a subset T of S such that every line in L intersects an odd number of elements of T?

Solution

The answer is any (m, n) such that $\lfloor m/2 \rfloor$ and $\lfloor n/2 \rfloor$ have the same parity; that is, both are even or both are odd.

First, we show these (m, n) work. It suffices to show that
(i) $(1, 4k+1)$, $(1, 4k+4)$, $(2, 4k+2)$, and $(2, 4k+3)$ work for any $k = 0, 1, 2, \ldots$,
(ii) (m, n) works if and only if (n, m) works, and
(iii) if (m, n) works, then $(m+2, n+2)$ works.

To show (i), we do $(1, 4k+4)$ by choosing T to contain $(1, i), (4k+5-i, 1), (i, 4k+5-i)$ for $1 \le i \le 2k+2$. We also obtain a solution to $(2, 4k+2)$ by using the same T for $2 \le i \le 2k+2$ instead. Then to do $(1, 4k+1)$, we choose T to contain $(1, i), (4k+2-i, 1), (i, 4k+2-i)$ for $1 \le i \le 2k$ and also throw in $(k+1, k+1), (2k+1, k+1)$, and $(k+1, 2k+1)$. We also obtain a solution to $(2, 4k+3)$ by using the same T for $2 \le i \le 2k$ instead (the three extra points are still included).

There are two edge cases that need to be covered separately due to the bounds on i. For $(m, n) = (1, 1)$, $T = \{(1, 1)\}$. For $(m, n) = (2, 3)$, $T = \{(1, 2), (1, 3), (1, 4), (2, 2), (3, 1), (4, 2)\}$.

(ii) is trivial, so it remains to show (iii). Given a subset T that works for (m, n), for $(m+2, n+2)$ we create a new subset T' such that for each (x, y) in T, T' contains $(x+2, y+2)$, and additionally T' also contains $(m+2, 1), (m+2, 2), (1, m+n+3), (2, m+n+2), (m+n+2, m+2), (m+n+3, m+2)$.

All that remains is to show that the remaining pairs (m, n) fail. Let M be a subset of L containing only those lines for which $x = 2k$ for some k, $y = 2k$ for some k, or $x+y = 2k+1$ for some k. All points in S are in exactly two lines of M, except those with both odd coordinates, which are contained in zero lines of M. Define

$$U = \{(t, l) \mid t \in T, \ell \in M, t \text{ lies on } \ell\}.$$

On one hand, for each point in T, there are an even number of lines in M containing that point, so U has an even number of elements. On the other hand, every line in M intersects an odd number of elements of T, so the parity of the number of elements in U matches that of M. This means that if (m, n) works, M must contain an even number of elements.

The total number of lines $x = 2k$ for some k is $\lfloor (m+n-1)/2 \rfloor$. There are the same number of lines $y = 2k$, so for the purposes of determining the parity of M, these two cancel out. The total number of lines $x + y = 2k + 1$ for some k is $\lfloor (m+n-1)/2 \rfloor$ if m is odd and else $\lfloor (m+n)/2 \rfloor$ if m is even. In both cases this total is equal to $\lfloor m/2 \rfloor + \lfloor n/2 \rfloor$, and this quantity is even if and only if $\lfloor m/2 \rfloor$ and $\lfloor n/2 \rfloor$ have the same parity.