USA Mathematical Talent Search

Round 1 Solutions

Year 35 — Academic Year 2023–2024

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1/1/35. Fill each unshaded cell of the grid with a number that is either 1, 3, or 5. For each cell, exactly one of the touching cells must contain the same number. Here touching includes cells that only share a point, i.e. touch diagonally.

1		1		3		5
			■			
■		1	1	3	5	
				■		■
	■					
5			5			

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

5	3	3	5	5	3	5
1	5	1	1	3	1	5
1	3	3	■	5	1	3
■	5	1	1	3	5	3
3	3	5	3	■	1	■
5	■	1	1	5	1	5
5	3	3	5	3	3	5

2/1/35. Suppose that the 101 positive integers

$$2024,\ 2025,\ 2026,\ \ldots,\ 2124$$

are concatenated in some order to form a 404-digit number. Can this number be prime?

Solution

No. This number can be written as

$$n = a_{100} \cdot 10{,}000^{100} + a_{99} \cdot 10{,}000^{99} + \cdots + a_1 \cdot 10{,}000^{1} + a_0 \cdot 10{,}000^{0}$$

for some rearrangement $a_0,\ a_1,\ \ldots,\ a_{100}$ of the positive integers $2024,\ 2025,\ \ldots,\ 2124$. Then

$$\begin{aligned}
n &\equiv a_{100} \cdot 10{,}000^{100} + a_{99} \cdot 10{,}000^{99} + \cdots + a_1 \cdot 10{,}000^{1} + a_0 \cdot 10{,}000^{0} \pmod{9999} \\
&\equiv a_{100} \cdot 1^{100} + a_{99} \cdot 1^{99} + \cdots + a_1 \cdot 1^{1} + a_0 \cdot 1^{0} \pmod{9999} \\
&\equiv a_{100} + a_{99} + \cdots + a_1 + a_0 \pmod{9999} \\
&\equiv 2024 + 2025 + \cdots + 2124 \pmod{9999} \\
&\equiv 101 \cdot 2023 + (1 + 2 + \cdots + 101) \pmod{9999} \\
&\equiv 101 \cdot 2023 + \frac{101 \cdot 102}{2} \pmod{9999} \\
&\equiv 101 \cdot (2023 + 51) \pmod{9999}.
\end{aligned}$$

Since $101 \mid 9999$, it follows that n must be a multiple of 101 and therefore cannot be prime.

3/1/35. Let $n \geq 2$ be a positive integer, and suppose buildings of height $1, 2, \ldots, n$ are built in a row on a street. Two distinct buildings are said to be *roof-friendly* if every building between the two is shorter than both buildings in the pair. For example, if the buildings are arranged $5, 3, 6, 2, 1, 4$, there are 8 roof-friendly pairs: $(5, 3)$, $(5, 6)$, $(3, 6)$, $(6, 2)$, $(6, 4)$, $(2, 1)$, $(2, 4)$, $(1, 4)$. Find, with proof, the minimum and maximum possible number of roof-friendly pairs of buildings, in terms of n.

Solution

The minimum is $n - 1$, as any adjacent pair of buildings is roof-friendly. The minimum is achieved by the arrangement $1, 2, 3, \ldots, n$.

The maximum is $2n-3$. We prove, by induction, the stronger result that an arrangement can have $2n-3$ roof-friendly pairs if and only if the two tallest buildings are at the two ends of the row of buildings. The base case $n = 2$ is trivial. For the inductive step, let $n > 2$ be a positive integer and assume the result is proved for all smaller n.

Suppose that the tallest building is in spot k with $1 < k < n$. Then there can be no roof-friendly pair where the two buildings are on the opposite sides of the tallest building. So by the inductive hypothesis there are at most $2k - 3$ roof-friendly pairs among the first k buildings and $2(n - k + 1) - 3$ roof-friendly pairs among the last $n - k + 1$ buildings, for a total of at most $2k - 3 + 2(n - k + 1) - 3 = 2n - 4$ pairs, below the desired maximum.

If the tallest building is at one end (say, in the leftmost space), then suppose that the second-tallest building is in space k with $1 < k < n$. The exact same argument as the above case shows that there are at most $2n - 4$ roof-friendly pairs.

Now assume that the tallest building is at one end (in space 1) and the second-tallest building is at the other end (in space n). Say that the third-tallest building is in space k with $1 < k < n$. Then by the inductive hypothesis there are exactly $2k - 3$ roof-friendly pairs among the first k buildings, exactly $2(n - k + 1) - 3$ roof-friendly pairs among the last $n - k + 1$ buildings, and the two tallest buildings are also a roof-friendly pair. Thus, there are $2k - 3 + 2(n - k + 1) - 3 + 1 = 2n - 3$ roof-friendly pairs, completing the proof.

4/1/35. Prove that, for any real numbers $1 \le \sqrt{x} \le y \le x^2$, the following system of equations has a real solution (a, b, c):

$$a+b+c = \frac{x+x^2+x^4+y+y^2+y^4}{2}$$
$$ab+ac+bc = \frac{x^3+x^5+x^6+y^3+y^5+y^6}{2}$$
$$abc = \frac{x^7+y^7}{2}.$$

Solution

The expressions on the left sides of the equations remind us of Vieta's formulas for cubic polynomials. (a, b, c) is a solution if and only if a, b, c are roots of the polynomial

$$P(z) = \frac{Q(z)+R(z)}{2}$$

where

$$Q(z) = (z-x)(z-x^2)(z-x^4)$$

and

$$R(z) = (z-y)(z-y^2)(z-y^4).$$

By symmetry, we can assume without loss of generality that $y \ge x$. Then, $1 \le x \le y \le x^2 \le y^2 \le x^4 \le y^4$. We consider several cases.

If $y > x$ and $y \neq x^2$, we have

$$Q(x) = 0, Q(y) > 0, Q(x^2) = 0, Q(y^2) < 0, Q(x^4) = 0, Q(y^4) > 0,$$
$$R(x) < 0, R(y) = 0, R(x^2) > 0, R(y^2) = 0, R(x^4) < 0, R(y^4) = 0.$$

Thus, we have

$$P(x) < 0, P(y) > 0,$$
$$P(x^2) > 0, P(y^2) < 0,$$
$$P(x^4) < 0, P(y^4) > 0.$$

Thus, P has roots in the intervals $(x, y), (x^2, y^2)$, and (x^4, y^4), so P has three real roots.

If $y > x$ and $y = x^2$, we have $Q(x^2) = R(x^2) = 0$, so $P(x^2) = 0$. We also have $Q(x^4) = R(x^4) = 0$, giving us $P(x^4) = 0$. So, x^2 and x^4 are two real roots. Substituting $y = x^2$ into $R(z)$, we obtain

$$P(x) = \frac{(z-x)(x-x^2)(z-x^4)+(z-x^2)(z-x^4)(z-x^8)}{2} = \frac{(z-x^2)(z-x^4)(2z-x^8-x)}{2}.$$

The third real root occurs when $2z - x^8 - x = 0$, which gives us $z = \dfrac{x^8 + x}{2}$ for the third real root.

If $1 < x = y$, then

$$Q(x) = 0, Q(y) = 0, Q(x^2) = 0, Q(y^2) = 0, Q(x^4) = 0, Q(y^4) = 0,$$
$$R(x) = 0, R(y) = 0, R(x^2) = 0, R(y^2) = 0, R(x^4) = 0, R(y^4) = 0.$$

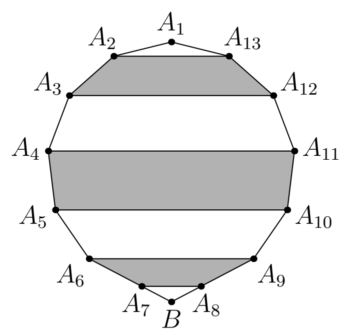
This gives us

$$P(x) = 0, P(x^2) = 0, P(x^4) = 0,$$

so x, x^2, and x^4 are three real roots.

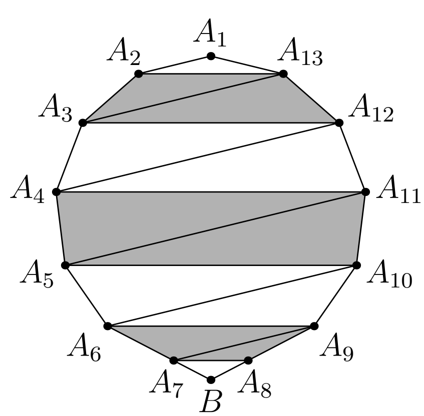
Finally, if $1 = x = y$, then $(a, b, c) = (1, 1, 1)$ is a solution to our system of equations.

5/1/35. Let $A_1A_2A_3\cdots A_{13}$ be a regular 13-gon, and let lines A_6A_7 and A_8A_9 intersect at B. Show that the shaded area below is half the area of the entire polygon (including triangle A_7A_8B).



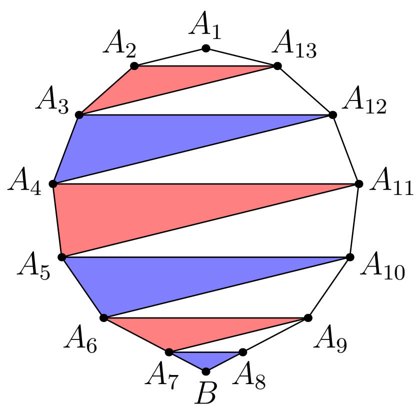
Solution

We want to show that the shaded portions and the unshaded portions of the polygon have the same area. We add the following lines to the diagram, splitting the polygon into triangles.

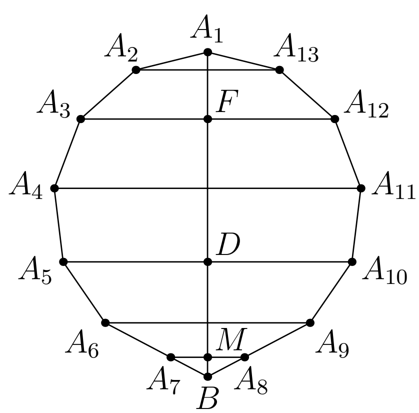


Triangles $A_1A_2A_{13}$ and $A_8A_7A_9$ are congruent, so their areas cancel. Likewise, triangles $A_3A_{12}A_{13}$ and $A_6A_{10}A_9$ are congruent, and triangles $A_4A_{11}A_{12}$ and $A_5A_{11}A_{10}$ are congruent.

Thus, the problem now is to prove that the total area of the red region is equal to the total area of the blue region.



In our next steps, we will compare each colored triangle to a triangle with base $\overline{A_7A_8}$. Let M be the midpoint of $\overline{A_7A_8}$, let D be the midpoint of $\overline{A_5A_{10}}$, and let F be the midpoint of $\overline{A_3A_{12}}$.



Triangles $A_2A_3A_{13}$ and $A_7A_8A_5$ are congruent, so they have the same area. And $\overline{A_5D}$ is parallel to $\overline{A_7A_8}$, so triangle $A_7A_8A_5$ has the same area as triangle A_7A_8D, which is

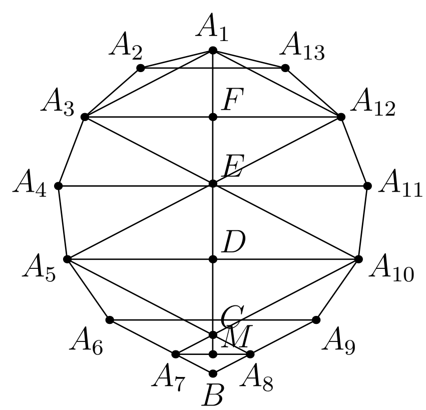
$$\frac{1}{2} \cdot s \cdot MD,$$

where $s = A_7A_8$. The area of triangle $A_6A_7A_9$ is also $\frac{1}{2} \cdot s \cdot MD$.

Similarly, we can show that the areas of triangles $A_3A_4A_{12}$ and $A_5A_6A_{10}$ are equal to $\frac{1}{2} \cdot s \cdot MF$, and the area of triangle $A_4A_5A_{11}$ is equal to $\frac{1}{2} \cdot s \cdot MA_1$. Now we must deal with the area of triangle A_7A_8B.

Let C be the intersection of diagonals $\overline{A_5A_8}$ and $\overline{A_7A_{10}}$, and let E be the intersection of diagonals $\overline{A_5A_{12}}$ and $\overline{A_3A_{10}}$. By symmetry, C and E lie on $\overline{A_1B}$.

To make it easier to refer to the angles in the diagram, let $\theta = \frac{180^\circ}{13}$. Each interior angle of the 13-gon is 11θ, so $\angle A_6A_7A_8 = \angle A_7A_8A_9 = 11\theta$. Then $\angle BA_7A_8 = \angle BA_8A_7 = 180^\circ - 11\theta = 2\theta$. Also, $\angle A_5A_8A_7 = \angle A_{10}A_7A_8 = 2\theta$. (For example, if O is the center of the 13-gon, then by the Inscribed Angle Theorem, $\angle A_5A_8A_7 = \frac{\angle A_5OA_7}{2} = \frac{4\theta}{2} = 2\theta$.) Hence, C is the reflection of B in $\overline{A_7A_8}$, which means triangle A_7A_8B has the same area as triangle A_7A_8C, which is $\frac{1}{2} \cdot s \cdot MC$.



The total area of the red triangles is then

$$2 \cdot \frac{1}{2} \cdot s \cdot MD + \frac{1}{2} \cdot s \cdot MA_1,$$

and the total area of the blue triangles is

$$2 \cdot \frac{1}{2} \cdot s \cdot MF + \frac{1}{2} \cdot s \cdot MC.$$

The problem now is to show that

$$2 \cdot MD + MA_1 = 2 \cdot MF + MC.$$

In the diagram, we can compute that $\angle A_8A_5A_{10} = \angle A_7A_{10}A_5 = \angle A_{12}A_5A_{10} = \angle A_3A_{10}A_5 = A_{10}A_3A_{12} = \angle A_5A_{12}A_3 = \angle A_1A_3A_{12} = \angle A_3A_{12}A_1 = 2\theta$. Hence, $A_5CA_{10}E$ and $A_3EA_{12}A_1$ are rhombi.

That means D is the midpoint of $\overline{CE}$, so $2 \cdot MD = MC + ME$. Also, F is the midpoint of $\overline{EA_1}$, so $2 \cdot MF = ME + MA_1$. Hence,

$$\begin{aligned} 2 \cdot MD + MA_1 &= MC + ME + MA_1 \\ &= 2 \cdot MF + MC, \end{aligned}$$

as desired.