USA Mathematical Talent Search

Round 3 Solutions

Year 34 — Academic Year 2022–2023

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1/3/34. In the 8×8 grid below, label 8 squares with X and 8 squares with Y such that:

1. No square can be labeled with both an X and a Y.
2. Each row and each column must contain exactly one square labeled X and one square labeled Y.
3. Any square marked with a $\star$ or a $\heartsuit$ cannot be labeled with an X or a Y.
4. We say that a square marked with a $\star$ or a $\heartsuit$ *sees* a label (X or Y) if one can move in a straight line horizontally or vertically from the marked square to the square with the label, without crossing any other squares with X's or Y's. It is OK to cross other squares marked with a $\star$ or $\heartsuit$. Using this definition:

   (a) Each square marked with a $\star$ must see exactly 2 X's and 1 Y.

   (b) Each square marked with a $\heartsuit$ must see exactly 1 X and 2 Y's.

		$\star$	$\star$	$\star$	$\star$	$\star$	
							$\star$
							$\star$
	$\heartsuit$						$\star$
							$\star$
							$\heartsuit$
				$\star$			

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution

X		⋆	⋆	⋆	⋆	⋆	Y
	Y	X					⋆
		Y	X				⋆
	♡		Y		X		⋆
					Y	X	⋆
				X		Y	♡
Y	X			⋆			
				Y			X

2/3/34. Let $\mathbb{Z}^+$ denote the set of positive integers. Determine, with proof, if there exists a function $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ such that

$$f(f(f(f(f(n))))) = 2022n$$

for all positive integers n.

Solution

Such a function can be constructed as follows. Let p be any prime that does not divide 2022, and let $\nu_p(n)$ denote the highest power of p that divides n: in particular $\nu_p(n)$ is the unique nonnegative integer k such that $p^k | n$ but $p^{k+1} \nmid n$. Then we may define f as

$$f(n) = \begin{cases} pn & \text{if } \nu_p(n) \not\equiv 4 \pmod 5, \\ \frac{2022n}{p^4} & \text{if } \nu_p(n) \equiv 4 \pmod 5. \end{cases}$$

Then applying f five times will cycle the quantity $\nu_p(f^i(n))$, in order, through all five equivalence classes modulo 5: four of the five applications of f will multiply by p, and the fifth will multiply by $\frac{2022}{p^4}$. (Since $p \nmid 2022$, dividing by p^4 will change the equivalence class modulo 5 of $\nu_p(f^i(n))$ from 4 to 0.) It does not matter in what order these equivalence classes get cycled through—the net result is multiplication by 2022, as desired.

(Note: This is not the only possible solution—other functions f exhibiting the desired property certainly exist. But the burden is on the solver to prove that their function works.)

3/3/34. A positive integer N is called *googolicious* if there are exactly 10^{100} positive integers x that satisfy the equation

$$\left\lfloor \frac{N}{\left\lfloor \frac{N}{x} \right\rfloor} \right\rfloor = x,$$

where $\lfloor z \rfloor$ denotes the greatest integer less than or equal to z. Find, with proof, all googolicious integers.

Solution

We claim that the answer is all N such that $25 \cdot 10^{198} + 5 \cdot 10^{99} \le N \le 25 \cdot 10^{198} + 10^{100}$.

Observe that, for any given x, we can uniquely write $N = qx + r$ where q, r are positive integers such that $0 \le r < x$. (This is the division $N \div x$ with quotient q and remainder r.) Then

$$\left\lfloor \frac{N}{\left\lfloor \frac{N}{x} \right\rfloor} \right\rfloor = \left\lfloor \frac{qx+r}{\left\lfloor \frac{qx+r}{x} \right\rfloor} \right\rfloor = \left\lfloor \frac{qx+r}{q} \right\rfloor = x + \left\lfloor \frac{r}{q} \right\rfloor.$$

So x satisfies the given equation if and only if $\left\lfloor \frac{r}{q} \right\rfloor = 0$, which occurs if and only if $r < q$.

If $x \le \lfloor \sqrt{N} \rfloor$, then since $\left(\lfloor \sqrt{N} \rfloor\right)^2 \le N$, we must have $q \ge \lfloor \sqrt{N} \rfloor \ge x > r$, so by the result above, x is a solution to the given equation. This gives $\sqrt{N}$ solutions to the given equation.

Also, for each $1 \le q \le \lfloor \sqrt{N} \rfloor$, we can consider the division $N \div q$ to conclude that there are unique nonnegative integers x and r with $0 \le r < q$ such that $N = qx + r$. This x is a solution to the given equation, and since $q \le \lfloor \sqrt{N} \rfloor$ and $\left(\lfloor \sqrt{N} \rfloor\right)^2 \le N$, we must have $x \ge \lfloor \sqrt{N} \rfloor$. This gives another $\sqrt{N}$ solutions to the given equation.

In summary, there are

$$\lfloor \sqrt{N} \rfloor \text{ solutions with } x \le \lfloor \sqrt{N} \rfloor \text{ and } q \ge \lfloor \sqrt{N} \rfloor,$$

and

$$\lfloor \sqrt{N} \rfloor \text{ solutions with } x \ge \lfloor \sqrt{N} \rfloor \text{ and } q \le \lfloor \sqrt{N} \rfloor.$$

These two solution sets overlap if and only if $x = \lfloor \sqrt{N} \rfloor$ and $q = \lfloor \sqrt{N} \rfloor$ works for both. This occurs when

$$N = \left(\lfloor \sqrt{N} \rfloor\right)^2 + r$$

with $r < \lfloor\sqrt{N}\rfloor$, or, equivalently, $N - \left(\lfloor\sqrt{N}\rfloor\right)^2 < \lfloor\sqrt{N}\rfloor$. That is,

$$\text{number of solutions for } N = \begin{cases} 2\lfloor\sqrt{N}\rfloor & \text{if } N \geq \left(\lfloor\sqrt{N}\rfloor\right)\left(\lfloor\sqrt{N}\rfloor+1\right), \\ 2\lfloor\sqrt{N}\rfloor - 1 & \text{if } N < \left(\lfloor\sqrt{N}\rfloor\right)\left(\lfloor\sqrt{N}\rfloor+1\right). \end{cases}$$

We want to choose N so that there are 10^{100} solutions, so we need N to satisfy $\lfloor\sqrt{N}\rfloor = 5\cdot 10^{99}$ and $N \geq (5\cdot 10^{99})(5\cdot 10^{99}+1)$. The first of these conditions gives $25\cdot 10^{198} \leq N \leq 25\cdot 10^{198} + 10^{100}$, and then applying the second condition gives our final answer of

$$25\cdot 10^{198} + 5\cdot 10^{99} \leq N \leq 25\cdot 10^{198} + 10^{100}.$$

4/3/34. Let ω be a circle with center O and radius 10, and let H be a point such that $OH = 6$. A point P is called *snug* if, for all triangles ABC with circumcircle ω and orthocenter H, we have that P lies on $\triangle ABC$ or in the interior of $\triangle ABC$. Find the area of the region consisting of all snug points.

Solution

We claim that the desired region is the ellipse with foci O and H and major axis of length 10. Call this ellipse $\mathcal{E}$. We will prove that P is snug if and only if P lies on or in the interior of $\mathcal{E}$.

Lemma: If $\triangle ABC$ has circumcenter O and orthocenter H, and A' is the point (other than A) where the line through AH meets the circumcircle of $\triangle ABC$, then $\overline{BC}$ is the perpendicular bisector of $\overline{A'H}$.

Proof of Lemma: Using the fact that $A'BAC$ is cyclic, we have $\angle A'AC = \angle A'BC$. Let F_A and F_B be the feet of the altitudes from A and B to $\overline{BC}$ and $\overline{AC}$, respectively. Then $\triangle AHF_B \sim \triangle BHF_A$, so $\angle HBC = \angle A'AC = \angle A'BC$. Thus $\triangle BF_AH \cong \triangle BF_AA'$, so $F_AH = F_AA'$, and the result follows. $\square$

Let P_A be the point where $\overline{OA'}$ intersects $\overline{BC}$. Note that $OP_A + P_AH = OP_A + P_AA' = OA' = 10$, and moreover, if X is any other point on $\overline{BC}$, then $OX + XH = OX + XA' > OA' = 10$. In particular, $\mathcal{E}$ is tangent to $\overline{BC}$ at P_A.

By the same reasoning, we see that $\mathcal{E}$ is tangent to all three sides of $\triangle ABC$, so that $\mathcal{E}$ (and its interior) lies entirely on or inside $\triangle ABC$. This is true for any triangle ABC with the given properties, so all points in $\mathcal{E}$ are snug.

Conversely, let Q be a point outside of $\mathcal{E}$, and let P be the point where $\overline{OQ}$ intersects $\mathcal{E}$. Then there is a triangle ABC with the given properties such that $\mathcal{E}$ is tangent to one of the sides of $\triangle ABC$ at P. Since $\overline{OQ}$ crosses this side, and O is inside of $\triangle ABC$, we conclude that Q is outside of $\triangle ABC$ and thus is not snug.

This proves the claim that the locus of snug points is $\mathcal{E}$ and its interior. Finally, $\mathcal{E}$ has a semi-major axis of length 5 and a semi-minor axis of length $\sqrt{5^2 - 3^2} = 4$, so its area is $\boxed{20\pi}$.

5/3/34. A lattice point is a point on the coordinate plane with integer coefficients. Prove or disprove: there exists a finite set S of lattice points such that for every line ℓ in the plane with slope 0, 1, -1, or undefined, either ℓ and S intersect at exactly 2022 points, or they do not intersect.

Solution

We claim that we can replace 2022 with any positive integer $k > 2$ in the problem statement, and construct a set S for that k, as follows. For every quadruple of integers (a, b, c, d) with $0 \le a, b, c, d < k$, we include in S the point

$$(a + kb + 2k^2c, -a + kb + 2k^3d).$$

Note that this gives k^4 distinct points in S: the quantities $a \pm kb$ each span the residue classes modulo k^2, and as (c, d) vary we are adding multiples of k^2 to each coordinate.

If ℓ is a vertical line given by $x = n$, then points in $\ell \cap S$ satisfy $a + kb + 2k^2c = n$. But such an (a, b, c), if it exists, is uniquely determined: (a, b) must satisfy $a + kb \equiv n \pmod{k^2}$, and then $c = \frac{n-(a+kb)}{2k^2}$. Hence, if such a solution (a, b, c) exists, we get exactly k points in $\ell \cap S$ by taking (a, b, c, d) as d varies among $\{0, \dots, k-1\}$.

If ℓ is a horizontal line given by $y = n$, then points in $\ell \cap S$ satisfy $-a + kb + 2k^3d = n$. But such an (a, b, d), if it exists, is uniquely determined: (a, b) must satisfy $-a + kb \equiv n \pmod{k^2}$, and then $d = \frac{n-(-a+kb)}{2k^3}$. Hence, if such a solution (a, b, d) exists, we get exactly k points in $\ell \cap S$ by taking (a, b, c, d) as c varies among $\{0, \dots, k-1\}$.

If ℓ is a line with slope 1 given by $x - y = n$, then points in $\ell \cap S$ satisfy $(a + kb + 2k^2c) - (-a + kb + 2k^3d) = n$, which simplifies to $2(a + k^2c - k^3d) = n$. But such an (a, c, d), if it exists, is uniquely determined: (a, c) must satisfy $a + k^2c \equiv \frac{n}{2} \pmod{k^3}$, and then $d = -\frac{\frac{n}{2}-(a+k^2c)}{k^3}$. Hence, if such a solution (a, c, d) exists, we get exactly k points in $\ell \cap S$ by taking (a, b, c, d) as b varies among $\{0, \dots, k-1\}$.

If ℓ is a line with slope -1 given by $x + y = n$, then points in $\ell \cap S$ satisfy $(a + kb + 2k^2c) + (-a + kb + 2k^3d) = n$, which simplifies to $2(kb + k^2c + k^3d) = n$. If such a (b, c, d) exists, it is uniquely given as the solution to $b + kc + k^2d \equiv \frac{n}{2k} \pmod{k^3}$. And if so, we get exactly k points in $\ell \cap S$ by taking (a, b, c, d) as a varies among $\{0, \dots, k-1\}$.

Hence for lines ℓ with slope 0, 1, -1, or undefined, we have shown that $\ell \cap S$ is either empty or consists of exactly k points.

Note that, more generally, we can construct S as

$$S = \{(f(a, b, c, d), g(a, b, c, d)) \mid 0 \le a, b, c, d < k\}$$

using functions f and g with the following properties:

1. Each (a, b, c, d) gives a unique point in S.
2. The equation $f(a, b, c, d) = n$ has either 0 or k solutions for any n.
3. The equation $g(a, b, c, d) = n$ has either 0 or k solutions for any n.
4. The equation $f(a, b, c, d) + g(a, b, c, d) = n$ has either 0 or k solutions for any n.
5. The equation $f(a, b, c, d) - g(a, b, c, d) = n$ has either 0 or k solutions for any n.

Problems 1-4 by USAMTS staff. Problem 5 appeared in another form on Tournament of Towns 2020. The authors were Nikolai Beluhov, Atanas Dinev, and Kostadin Garov.