USA Mathematical Talent Search
Round 1 Solutions
Year 34 — Academic Year 2022–2023
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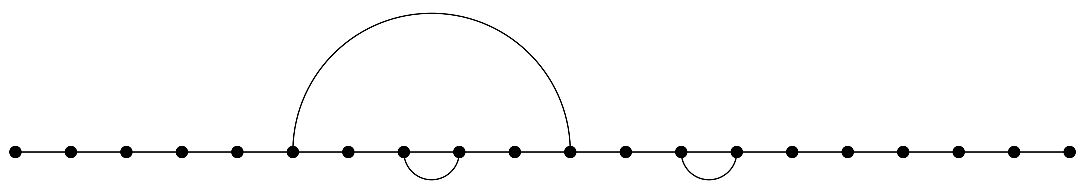
1/1/34. Shown is a segment of length 19, marked with 20 points dividing the segment into 19 segments of length 1. Draw 20 semicircular arcs, each of whose endpoints are two of the 20 marked points, satisfying all of the following conditions:

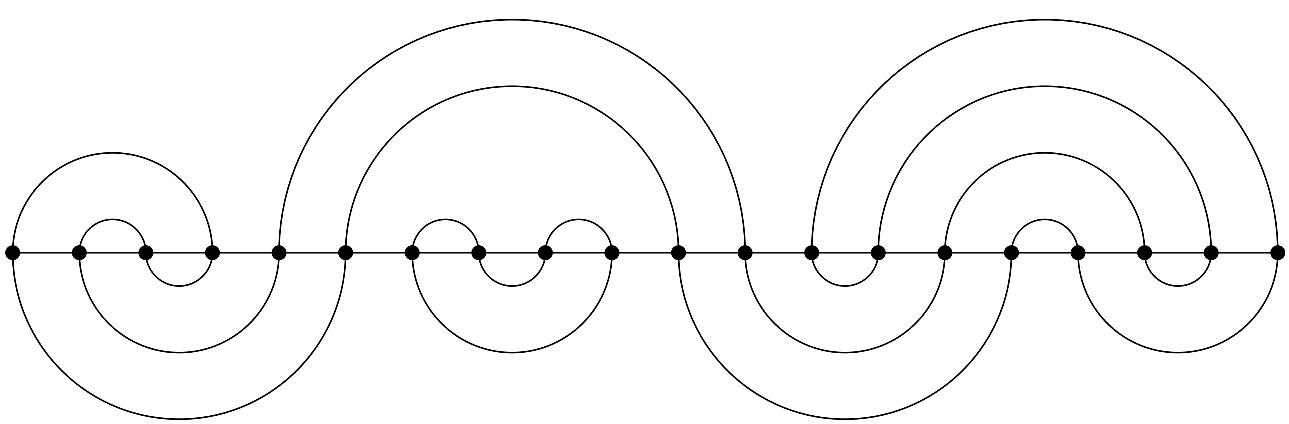
1. When the drawing is complete, there will be:
    - 8 arcs with diameter 1,
    - 6 arcs with diameter 3,
    - 4 arcs with diameter 5,
    - 2 arcs with diameter 7.
2. Each marked point is the endpoint of exactly two arcs: one above the segment and one below the segment.
3. No two distinct arcs can intersect except at their endpoints.
4. No two distinct arcs can connect the same pair of points. (That is, there can be no full circles.)

Three arcs have already been drawn for you.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/1/34. Given a sphere, a *great circle* of the sphere is a circle on the sphere whose diameter is also a diameter of the sphere. For a given positive integer n, the surface of a sphere is divided into several regions by n great circles, and each region is colored black or white. We say that a coloring is *good* if any two adjacent regions (that share an arc as boundary, not just a finite number of points) have different colors. Find, with proof, all positive integers n such that in every good coloring with n great circles, the sum of the areas of the black regions is equal to the sum of the areas of the white regions.

Solution

The answer is all odd n.

If n is odd, consider any point P on the sphere, not on one of the given great circles, and let Q be the point directly opposite on the sphere (so that $\overline{PQ}$ is a diameter of the sphere). Consider any great circle that contains P and Q. If we imagine traveling from P to Q along the new great circle, we will cross each of the n original great circles exactly once. At each crossing the color will switch from black to white or vice versa. In particular the color will switch n times, and since n is odd, this means that Q will have the opposite color from P. Hence, every point and its opposite point will have opposite colors, and thus the total areas of the black and white regions will be equal.

On the other hand, if n is even then it is possible for a coloring to have unequal black and white areas. For instance, let N and S be opposite points on the sphere, and let all n great circles pass through N and S. (Think of N and S as the north and south poles on a globe, and the great circles as arcs of constant longitude.) Then the circles can be rotations about the segment $\overline{NS}$ arbitrarily, and in particular the black or white areas can be made arbitrarily large or small. For instance, if $n = 2$, then if the two great circles are an angle of $30°$ apart from each other, then one color will be $\frac{1}{6}$ of the sphere and the other color will be $\frac{5}{6}$ of the sphere.

3/1/34. Prove that there is a unique 1000-digit number N in base 2022 with the following properties:

1. All of the digits of N (in base 2022) are 1's or 2's, and
2. N is a multiple of the base-10 number 2^{1000}.

(Note that you must prove both that such a number exists and that there is not more than one such number. You do not have to write down the number! In fact, please don't!)

Solution

We prove the more general result that there is a unique n-digit number N in base 2022 that is a multiple of 2^n, for all positive n, by induction. If $n = 1$ then the base case is trivial as 2 is a multiple of $2^1 = 2$ but 1 is not.

For the induction step, suppose the result holds for all $n < k$ for some $k \geq 2$, and let M be, by inductive hypothesis, the unique $(k-1)$-digit number with digits all 1's and 2's that is a multiple of 2^{k-1}. This means that we can write $M = m \cdot 2^{k-1}$ for some positive integer m. To prove that N exists, we will show that we can choose a digit d, either 1 or 2, to append to the front of M such that $N = d \cdot (2022)^{k-1} + M$ is a multiple of 2^k. Indeed, we can write N as

$$N = d \cdot (1011)^{k-1} \cdot 2^{k-1} + m \cdot 2^{k-1} = \big(d \cdot (1011)^{k-1} + m\big) \cdot 2^{k-1},$$

hence N is a multiple of 2^k if and only if $d \cdot (1011)^{k-1} + m$ is even. If m is odd, we can choose $d = 1$ to make this quantity even, and if m is even we can choose $d = 2$ to make this quantity even. Further note that in each of these cases this is the only choice of $d \in \{1, 2\}$ that works.

To show that N is unique, observe that if N is a k-digit number that is a multiple of 2^k, then we can write $N = c \cdot 2^k$ for some positive integer n. Then, if we let N' be N with its initial digit d removed, then

$$N' = N - d \cdot (2022)^{k-1} = c \cdot 2^k - d \cdot (1011)^{k-1} \cdot 2^{k-1} = \big(2c - d \cdot (1011)^{k-1}\big) \cdot 2^{k-1}$$

is a multiple of 2^{k-1}. Since this $(k-1)$-digit number is unique by the inductive hypothesis, we conclude that the unique N that satisfies the condition is the one that is constructed by appending a 1 or 2 to a $(k-1)$-digit number that satisfies the condition, as described above.

4/1/34. Grogg and Winnie are playing a game using a deck of 50 cards numbered 1 through 50. They take turns with Grogg going first. On each turn a player chooses a card from the deck—this choice is made deliberately, *not* at random—and then adds it to one of two piles (both piles are empty at the start of the game). After all 50 cards are in the two piles, the values of the cards in each pile are summed, and Winnie wins the positive difference of the sums of the two piles, in dollars. (For instance, if the first pile has cards summing to 510 and the second pile has cards summing to 765, then Winnie wins $255.) Winnie wants to win as much as possible, and Grogg wants Winnie to win as little as possible. If they both play with perfect strategy, find (with proof) the amount that Winnie wins.

Solution

Define the "score" to be the positive difference between the piles at any point in the game; note that Winnie wins the final score amount in dollars.

We claim that Winnie has a strategy that guarantees that she will win at least $75, no matter how Grogg plays. Her strategy is:

1. Make sure that 50 and 49 end up in the same pile: whenever Grogg plays one of these, she immediately plays the other in the same pile.
2. Make sure that $2k-1$ and $2k$ end up in opposite piles, for all $1 \leq k \leq 24$: whenever Grogg plays one of these, she immediately plays the other in the other pile.

(a) ensures that 99 gets added to one of the piles, and (b) ensures each of the 24 pairs $\{2k-1, 2k\}$ changes the score by ± 1. (The order in which these occur does not matter—we can imagine (a) happening at the start of the game.) Thus, at worst, Winnie will win $99 - 24 = 75$ dollars.

On the other hand, we claim that Grogg has a strategy that limits Winnie's winnings to at most $75, no matter how Winnie plays. Grogg's strategy is to always play the largest remaining card to the pile with the lower total. (If the piles are equal it does not matter which pile he plays to.) Using this strategy, if the score is s_i at the start of Grogg's i^{th} turn (where $1 \leq i \leq 25$), and the maximum card remaining in the deck is m_i, then before his next turn, the score will be at most the larger of $2m_i - 1$ or $s_i - 1$. That is, $s_{i+1} \leq \max\{2m_i, s_i\} - 1$. But $m_i \leq 51 - i$ if Grogg follows his strategy of always playing the largest card. So it can be verified by induction that $s_{i+1} \leq 100 - i$ for all i, and therefore at the end of the game $s_{26} \leq 75$ is the upper bound for the final score under Grogg's strategy.

Therefore, if both players play optimally, the final score will be 75. Note that using the above strategies, one pile will be $\{1, 3, 5, 7, \ldots, 49, 50\}$, which sums to 675, and the other pile will be $\{2, 4, 6, 8, \ldots, 48\}$, which sums to 600.

Source Note: Based on Problem 142 from *Elements of Mathematics, Book B: EM Problem Book*, Cermel, Inc., 1975.

5/1/34. We call a positive integer n *sixish* if $n = p(p+6)$, where p and $p+6$ are prime numbers. For example, $187 = 11 \cdot 17$ is sixish, but $475 = 19 \cdot 25$ is not sixish.

Define a function f on positive integers such that $f(n)$ is the sum of the squares of the positive divisors of n. For example, $f(10) = 1^2 + 2^2 + 5^2 + 10^2 = 130$.

(a) Find, with proof, an irreducible polynomial function $g(x)$ with integer coefficients such that $f(n) = g(n)$ for all sixish n. ("Irreducible" means that $g(x)$ cannot be factored as the product of two polynomials of smaller degree with integer coefficients.)

(b) We call a positive integer n *pseudo-sixish* if n is not sixish but nonetheless $f(n) = g(n)$, where $g(n)$ is the polynomial function that you found in part (a). Find, with proof, all pseudo-sixish positive integers.

Solution

(a) Let $n = p(p+6)$ be sixish. The divisors of n are $\{1, p, p+6, n\}$, so

$$\begin{aligned} f(n) &= 1^2 + p^2 + (p+6)^2 + n^2 \\ &= 1 + p^2 + p^2 + 12p + 36 + n^2 \\ &= n^2 + 2p(p+6) + 37 \\ &= n^2 + 2n + 37. \end{aligned}$$

Thus $g(n) = n^2 + 2n + 37$.

(b) The answer is that 27 is the only psuedo-sixish number.

For any n, let $h(n) = g(n) - f(n)$. Note that n is pseudo-sixish if $h(n) = 0$ where n is not of the form $p(p+6)$ where p and $p+6$ are both prime.

First, note that $h(1) = 40 - 1 = 39$, so 1 is not pseudo-sixish.

Next, note that if n is prime, then $h(n) = (n^2 + 2n + 37) - (n^2 + 1) = 2n + 36 \neq 0$, so any prime number is not pseudo-sixish.

Similarly, if $n = p^2$ for some prime p, then $h(n) = (p^4 + 2p^2 + 37) - (p^4 + p^2 + 1) = p^2 + 36 \neq 0$, so any square of a prime number is not pseudo-sixish.

This means that if n is pseudo-sixish, then we can choose integers r, s with $1 < r < s$ such that $n = rs$. In this case, $f(n) = 1 + r^2 + s^2 + n^2 + x_n$, where x_n is the sum of the squares of all divisors of n other than $\{1, r, s, n\}$. Then

$$0 = h(n) = n^2 + 2n + 37 - (1 + r^2 + s^2 + n^2 + x_n) = 2n + 36 - r^2 - s^2 - x_n.$$

But $2n = 2rs$, so $2n - r^2 - s^2 = -(s-r)^2$, and we can rewrite the above equation as $x_n = 36 - (s-r)^2$.

If $x_n = 0$, then $s - r = 6$ and $\{1, r, s, n\}$ are the only divisors of n. One possibility is that r and s are both prime, but that means that n is sixish and not pseudo-sixish. The other possibility is that $n = r^3$ with r prime, so that $s = r^2$. But then $r^2 - r = 6$, so $r = 3$. Hence $n = 3^3 = 27$ is pseudo-sixish.

If $x_n > 0$, then since $x_n = 36 - (s-r)^2$ it must be one of $\{11, 20, 27, 32, 35\}$. But at the same time, x_n is the sum of squares of distinct divisors of n greater than 1 (other than r, s, and n itself). The only possibility is $x_n = 20 = 2^2 + 4^2$, so that the divisors of n are $\{1, 2, 4, r, s, n\}$. But if n has exactly 6 divisors, it must be of the form p^2q or p^5 for primes p and q. In either of these cases we must have $p = 2$, but then we have in the former case $r = q$ and $s = 2q$, and in the latter case we have $r = 8$ and $s = 16$, and in neither of these is $n = rs$ satisfied. Thus $x_n > 0$ is not possible.

Therefore, the only psuedo-sixish number is 27.

Source note: This problem and solution were inspired by:
T. Chaobankoh and P. Chomchit, "A Product of Two Primes with Difference 2," *American Mathematical Monthly*, 129(2), p. 115.