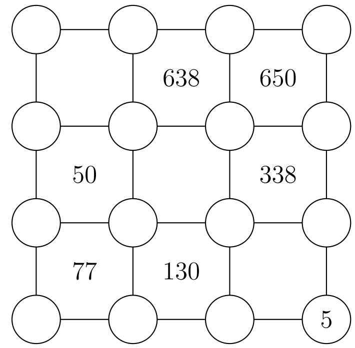
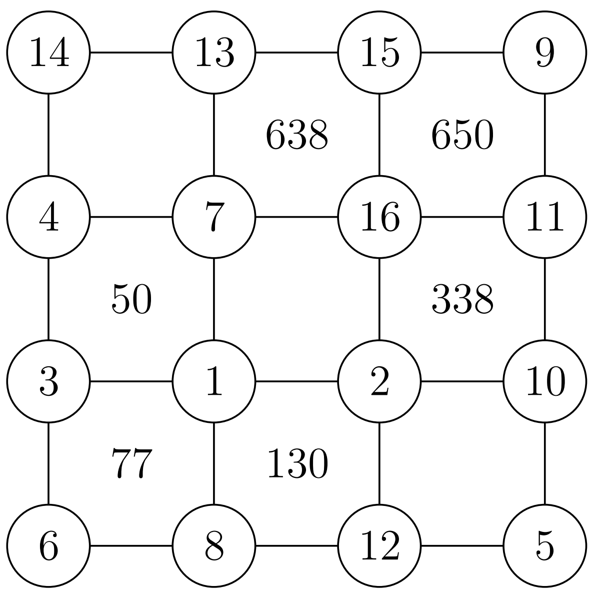
USA Mathematical Talent Search
Round 2 Solutions
Year 31 — Academic Year 2019–2020
www.usamts.org

1/2/31. Fill in each empty white circle with a number from 1 to 16 so that each number is used exactly once. One number has been given to you. If a square has a given number inside and its four vertices contain the numbers a, b, c, d in clockwise order, then the number inside the square must be equal to $(a + c)(b + d)$.



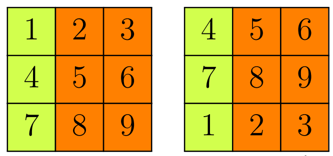
There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



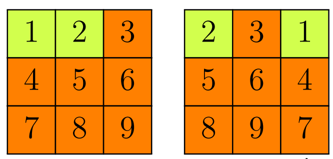
2/2/31. A 3×3 grid of blocks is labeled from 1 through 9. Cindy paints each block orange or lime with equal probability and gives the grid to her friend Sophia.

Sophia then plays with the grid of blocks. She can take the top row of blocks and move it to the bottom, as shown.



Grid A Grid A'

She can also take the leftmost column of blocks and move it to the right end, as shown.



Grid B Grid B'

Sophia calls the grid of blocks *citrus* if it is impossible for her to use a sequence of the moves described above to obtain another grid with the same coloring but a different numbering scheme. For example, Grid B is *citrus*, but Grid A is not *citrus* because moving the top row of blocks to the bottom results in a grid with a different numbering but the same coloring as Grid A.

What is the probability that Sophia receives a *citrus* grid of blocks?

Solution 1: Complementary Counting

As shown below, each grid of numbered blocks can be permuted in $3 \cdot 3 = 9$ different ways by block moves.

1	2	3
4	5	6
7	8	9

A

2	3	1
5	6	4
8	9	7

B

3	1	2
6	4	5
9	7	8

C

4	5	6
7	8	9
1	2	3

D

5	6	4
8	9	7
2	3	1

E

6	4	5
9	7	8
3	1	2

F

7	8	9
1	2	3
4	5	6

G

8	9	7
2	3	1
5	6	4

H

9	7	8
3	1	2
6	4	5

I

There are $2^9 = 512$ total colorings, and we count the grids that are not citrus. We start with a coloring for grid **A**. If grid **B** has the same coloring as grid **A**, then cells $1, 2, 3$ must all have the same color, cells $4, 5, 6$ must all have the same color, and cells $7, 8, 9$ must all have the same color. Continuing in this manner, we see that the not-citrus grids must look like one of the following four options, where a, b, c represent either orange or lime:

a	a	a
b	b	b
c	c	c

W

a	b	c
a	b	c
a	b	c

X

a	b	c
c	a	b
b	c	a

Y

a	b	c
b	c	a
c	a	b

Z

If any of **W**, **X**, **Y**, **Z** have the same coloring, then the whole grid must be entirely orange or entirely lime. There are 2 such colorings. Each of **W**, **X**, **Y**, **Z** has $2^3 - 2 = 6$ colorings that do not overlap with any of the other grids, so there are a total of $2 + 4 \cdot 6 = 26$ not-citrus colorings. So the probability that Sophia's grid of blocks is citrus is

$$\frac{512 - 26}{512} = \frac{486}{512} = \boxed{\frac{243}{256}}.$$

□

Solution 2: Group Theory

Let S be the set of 2^9 possible colorings of the grid, and let G be the group of block moves. G is the product of two cyclic groups of order 3, and the order of G is 9. The action of G on S partitions S into orbits of size $1, 3$, and 9. We seek the number of elements contained in an orbit of size 9, which corresponds to the number of citrus grids.

We can find the total number of orbits using Burnside's Lemma. The number of orbits is given by

$$\frac{1}{|G|} \sum_{g \in G} |\text{fix } g|.$$

The identity element of G fixes all 512 elements of S, and every other element of G fixes 2^3 elements of S (see the diagrams in Solution 1). So the number of orbits is $\frac{1}{9}(512+8\cdot 8) = 64$.

Elements with orbit size 1 are fixed by every element of G, and there are only 2 of these (when the grid is entirely orange or entirely lime). Let x denote the number of orbits of size 3 and let y denote the number of orbits of size 9. There are $512 - 2 = 510$ elements and $64 - 2 = 62$ orbits remaining, so

$$3x + 9y = 510,$$

$$x + y = 62.$$

Solving this system, we find $y = 54$. Therefore there are $54 \cdot 9 = 486$ elements in orbits of order 9 and thus 486 citrus grids of blocks. Therefore the probability that Sophia receives a citrus grid of blocks is $\frac{486}{512} = \boxed{\frac{243}{256}}$. □

3/2/31. Call a quadruple of positive integers (a, b, c, d) *fruitful* if there are infinitely many integers m such that $\gcd(am + b, cm + d) = 2019$. Find all possible values of $|ad - bc|$ over fruitful quadruples (a, b, c, d).

Solution

Since $2019 = \gcd(am + b, cm + d)$, we know that 2019 must divide

$$a(cm + d) - c(am + b) = ad - bc.$$

This means $|ad - bc| = 2019k$ for a nonnegative integer k. We claim all positive k are achievable by taking the fruitful quadruple $(k, 2019, k, 4038)$. (If $k = 0$, this quadruple isn't fruitful because $a = c = k = 0$.)

First, we see that this quadruple gives us

$$|ad - bc| = |k \cdot 4038 - k \cdot 2019| = 2019k.$$

Next, we verify that this quadruple is fruitful. We have

$$\gcd(am + b, cm + d) = \gcd(km + 2019, km + 4038) = \gcd(km + 2019, 2019).$$

If m is a multiple of 2019,

$$\gcd(km + 2019, 2019) = 2019.$$

There are infinitely many multiples of 2019, so $(k, 2019, k, 4038)$ is a fruitful quadruple as desired, and $|ad - bc|$ can be any positive multiple of 2019.

We now show that $|ad - bc| = 0$ cannot be achieved. Notice that $ad - bc = 0$ is equivalent to $\frac{a}{b} = \frac{c}{d}$. Then there exists some fraction in lowest terms $\frac{e}{f}$ such that $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Thus, there exist positive integers x and y such that $a = ex$, $b = fx$, $c = ey$, and $d = fy$.

We have

$$\begin{aligned}\gcd(am + b, em + f) &= \gcd(emx + fx, em + f)\\ &= \gcd(x(em + f), em + f)\\ &= em + f.\end{aligned}$$

Similarly, $\gcd(cm+d, em+f) = em+f$. Thus, $\gcd(am+b, cm+d)$ is a multiple of $em+f$. Since e, f, and m are positive integers, $em+f$ monotonically increases as m increases. Therefore, at some point $em+f > 2019$, and from that point on, we can no longer have $\gcd(am+b, cm+d) = 2019$. So, there are a finite number of integers m for which $\gcd(am+b, cm+d) = 2019$. Thus, if $|ad-bc| = 0$, then (a, b, c, d) cannot be a fruitful quadruple.

$|ad-bc|$ can be any positive multiple of 2019. $\square$

4/2/31. Princess Pear has 100 jesters with heights $1, 2, \ldots, 100$ inches. On day n with $1 \le n \le 100$, Princess Pear holds a court with all her jesters with height at most n inches, and she receives two candied cherries from every group of 6 jesters with a median height of $n-50$ inches. A jester can be part of multiple groups.

On day 101, Princess Pear summons all 100 jesters to court one final time. Every group of 6 jesters with a median height of 50.5 inches presents one more candied cherry to the Princess. How many candied cherries does Princess Pear receive in total?

Please provide a numerical answer (with justification).

Solution 1: Algebra

The third-shortest jester in any group must have a height of at least 3, so Princess Pear does not receive any candied cherries until day 54.

Given a group of jesters with median height $n-50$, let the third- and fourth-tallest jesters have heights $n-50+x$ and $n-50-x$ for some positive integer x. The fourth-tallest jester has height at least 3, so x can be at most $n-53$. Then there are $\binom{n-51-x}{2}$ ways to choose the two shortest jesters and $\binom{n-(n-50+x)}{2} = \binom{50-x}{2}$ ways to choose the two tallest jesters in the group. So in the first 100 days, Princess Pear receives

$$2\sum_{n=54}^{100}\sum_{x=1}^{n-53}\binom{n-51-x}{2}\binom{50-x}{2}$$

candied cherries.

On the 101st day, if the fourth-tallest jester in a group has height t for t between 3 and 50, the third-tallest jester must have height $101-t$. Then there are $\binom{t-1}{2}$ ways to choose the two shortest jesters and $\binom{100-(101-t)}{2} = \binom{t-1}{2}$ ways to choose the two tallest jesters in the group.

Therefore the total number of candied cherries received by the Princess is

$$\sum_{t=3}^{50}\binom{t-1}{2}^2 + 2\cdot\sum_{n=54}^{100}\sum_{x=1}^{n-53}\binom{n-51-x}{2}\binom{50-x}{2}.$$

Switching the order of summation and re-indexing, this is equivalent to

$$\sum_{t=2}^{49}\binom{t}{2}^2 + 2\cdot\sum_{x=1}^{47}\sum_{n=x+53}^{100}\binom{n-51-x}{2}\binom{50-x}{2}.$$

Re-indexing by replacing x in the second sum with $50-t$, we get

$$\sum_{t=2}^{49}\binom{t}{2}^2 + 2\cdot\sum_{t=3}^{49}\sum_{n=103-t}^{100}\binom{n+t-101}{2}\binom{t}{2}.$$

Re-indexing one more time by replacing $n+t-101$ with s, the desired sum is equivalent to

$$\sum_{t=2}^{49}\binom{t}{2}^2 + 2\cdot\sum_{t=3}^{49}\sum_{s=2}^{t-1}\binom{s}{2}\binom{t}{2}$$

$$= \left(\sum_{t=2}^{49}\binom{t}{2}\right)^2.$$

By the Hockey-Stick identity, this is

$$\binom{50}{3}^2 = \boxed{384,160,000}.$$

□

Solution 2: Bijection

We claim the answer is $\binom{50}{3}^2$, and to do that we find a bijection between candied cherries received by the Princess and 6-jester groups with 3 jesters of height of at most 50 inches and 3 jesters of height at least 51 inches.

Let $a_1 < a_2 < a_3 < a_4 < a_5 < a_6$ be the heights of a group of jesters with median height $n-50$. For the first candied cherry this group presents to the Princess, we instead collect the group of jesters with heights

$$\{a_1, a_2, a_3, 100-n+a_4, 100-n+a_5, 100-n+a_6\}.$$

We know that $a_1 < a_2 < a_3 \le 50$, and we also have $100-n+a_4 > 50 \Leftrightarrow a_4 > n-50$ because $n-50 = \frac{a_3+a_4}{2}$. The median height of the new group of jesters is $\frac{a_3+100-n+a_4}{2} = \frac{2n-100+100-n}{2} = \frac{n}{2}$, which is always at most 50.

For the second candied cherry that $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ presents to the Princess, we instead collect the group of jesters with heights

$$\{n+1-a_6, n+1-a_5, n+1-a_4, 101-a_3, 101-a_2, 101-a_1\}.$$

Because $a_3 \le 50$, we have $101-a_3 \ge 51$, and $n+1-a_4 \le 50$ because $a_4 > n-50$. The median height of this group is $\frac{n+1-a_4+101-a_3}{2} = \frac{n+102-2n+100}{2} = 101-\frac{n}{2}$, which is at least 51.

This accounts for all possible groups of jesters with 3 jesters of height at most 50 and 3 jesters of height at least 51, except for the groups with median height 50.5. Any group with median height 50.5 already has 3 jesters with height in $\{1, \dots, 50\}$ and 3 jesters with height in $\{51, \dots, 100\}$, so we don't have to change anything about these groups.

The process of transforming groups of jesters on day n into groups of jesters with 3 heights in $\{1, \dots, 50\}$ and 3 heights in $\{51, \dots, 100\}$ is reversible as long as we can recover n. We can do this based on the median height of the group, so we have our desired bijection. There are $\binom{50}{3}^2$ groups of 6 jesters with the desired property, so there are also

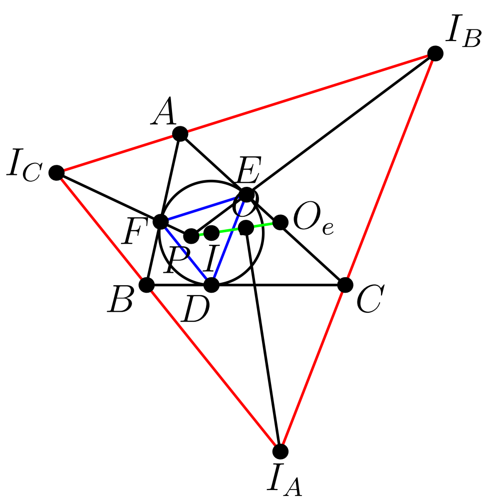
$$\binom{50}{3}^2 = \boxed{384,160,000}$$

candied cherries received by the Princess. $\square$

5/2/31. Let ABC be a triangle with circumcenter O, A-excenter I_A, B-excenter I_B, and C-excenter I_C. The incircle of $\triangle ABC$ is tangent to sides $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at D, E, and F respectively. Lines $\overline{I_B E}$ and $\overline{I_C F}$ intersect at P. If the line through O perpendicular to $\overline{OP}$ passes through I_A, prove that $\angle A = 60^\circ$.

An excenter is the point of concurrency among one internal angle bisector and two external angle bisectors of a triangle.

Solution



Let O_e be the circumcenter of $\triangle I_A I_B I_C$. $\overline{EF}$ and $\overline{I_B I_C}$ are both perpendicular to the internal angle bisector $\overline{AI_A}$ of $\angle BAC$, so $\overline{EF} \parallel \overline{I_B I_C}$. Similarly, $\overline{FD} \parallel \overline{I_C I_A}$ and $\overline{DE} \parallel \overline{I_A I_B}$. Thus, $\triangle DEF$ and $\triangle I_A I_B I_C$ are homothetic. The center of homothety is $\overline{I_B E} \cap \overline{I_C F} = P$.

The homothety centered at P carrying $\triangle DEF$ to $\triangle I_A I_B I_C$ must also carry the circumcenter I of $\triangle DEF$ to the circumcenter O_e of $\triangle I_A I_B I_C$, so P is collinear with I and O_e. I is the incenter of $\triangle ABC$, so it is also the orthocenter of $\triangle I_A I_B I_C$. Therefore $\overline{PIO_e}$ is the Euler line of $\triangle I_A I_B I_C$. Since O is the nine-point center of $\triangle I_A I_B I_C$, we know that O lies on the Euler line as well. Thus, $\overline{OP}$ is actually just the Euler line of $\triangle I_A I_B I_C$.

The problem statement gives us that $\overline{I_A O}$ is perpendicular to the Euler line $\overline{IO_e}$ of $\triangle I_A I_B I_C$. However, O is equidistant from I and O_e, so $\overline{I_A O}$ must be the perpendicular bisector of $\overline{IO_e}$. So $I_A O_e = I_A I = 2 IO_e \cos \angle I_B I_A I_C$, which occurs only when $\angle I_B I_A I_C = 60^\circ$. Then $\angle A = \angle BAC = 180^\circ - 2\angle I_B I_A I_C = 60^\circ$ as desired.

□