USA Mathematical Talent Search

Round 1 Solutions

Year 31 — Academic Year 2019–2020

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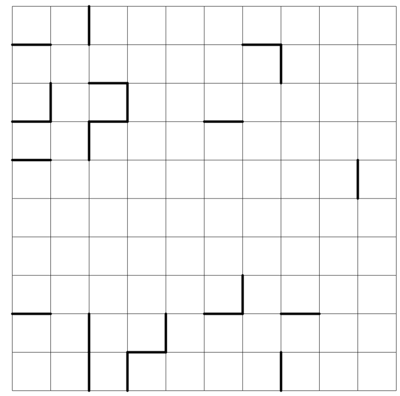
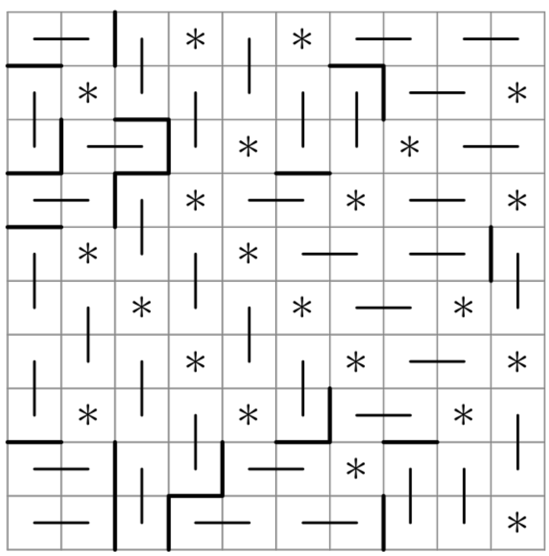
1/1/31. Partition the grid into 1 by 1 squares and 1 by 2 dominoes in either orientation, marking dominoes with a line connecting the two adjacent squares, and 1 by 1 squares with an asterisk (∗). No two 1 by 1 squares can share a side. A *border* is a grid segment between two adjacent squares that contain dominoes of opposite orientations. All borders have been marked with thick lines in the grid.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/1/31. Let x, y, and z be real numbers greater than 1. Prove that if $x^y = y^z = z^x$, then $x = y = z$.

Solution

It seems hard to make progress directly on this problem, so we try to approach it by contradiction. If $x = y = z$ is false, then maybe $x < y$. Let's see what goes wrong if we make this assumption.

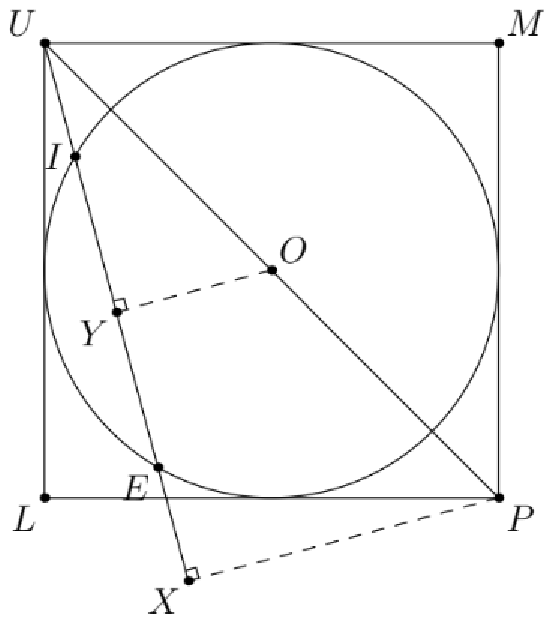
If $x < y$, then $x^y = y^z > x^z$. Since x, y, and z are all greater than 1, this means $y > z$. We also have $x^y = z^x < z^y$, so $x < z$. But from $y > z$ we get $y^z = z^x < y^x$, implying $z < x$.

We can't have $x < z$ and $z < x$ at the same time, so this is a contradiction and x cannot be less than y. By similar logic, we deduce that x cannot be greater than y either. This forces $x = y$. Similarly, $y = z$ and $x = y = z$ as desired. $\square$

3/1/31. Circle ω is inscribed in unit square $PLUM$, and points I and E lie on ω such that U, I, and E are collinear. Find, with proof, the greatest possible area for $\triangle PIE$.

Solution

Let O be the center of ω. Let X be the foot of the altitude from P to $\overline{IE}$, and let Y be the foot from O to $\overline{IE}$.



$\triangle UOY \sim \triangle UPX$ with $\frac{UP}{UO} = 2$, so $OY = \frac{PX}{2}$. $\triangle OIE$ and $\triangle PIE$ have the same base, IE, and the height OY of $\triangle OIE$ is half the height PX of $\triangle PIE$, so the area of $\triangle OIE$ is half of the area of $\triangle PIE$. It suffices to maximize the area of $\triangle OIE$.

The radius of ω is $\frac{1}{2}$, so by the sine area formula,

$$[OIE] = \frac{1}{2} OI \cdot OE \cdot \sin \angle IOE = \frac{\sin \angle IOE}{8}.$$

This takes a maximum value of $\frac{1}{8}$ when $\angle IOE = 90^\circ$, which would make the area of $\triangle PIE$ equal to $\boxed{\frac{1}{4}}$. It remains to prove that such a configuration is possible.

If $\angle IOE = 90^\circ$, then $IE = \frac{\sqrt{2}}{2}$. We need to find a location of I where this is true. We first find an equivalent condition on I, and then we show it is possible for this condition to hold.

Let $UI = x$, so $UE = x + \frac{\sqrt{2}}{2}$. The length of the tangent from U to ω is $\frac{1}{2}$, so by Power of a Point

$$UI \cdot UE = \left(\frac{1}{2}\right)^2$$

$$x^2 + \frac{\sqrt{2}}{2}x - \frac{1}{4} = 0$$

By the quadratic formula,

$$x = \frac{1}{2}\left(-\frac{\sqrt{2}}{2} + \sqrt{\frac{1}{2} - 4\left(-\frac{1}{4}\right)}\right),$$

where we have discarded the negative solution because length cannot be negative. This simplifies to

$$x = -\frac{\sqrt{2}}{4} + \frac{1}{2}\sqrt{\frac{3}{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}.$$

All of these steps are reversible, so as long as we can find a location of I satisfying $UI = \frac{\sqrt{6}-\sqrt{2}}{4}$ and a location of E satisfying $UE = \frac{\sqrt{6}-\sqrt{2}}{4} + \frac{\sqrt{2}}{2} = \frac{\sqrt{6}+\sqrt{2}}{4}$, we can obtain $[PIE] = \frac{1}{4}$.

We just need to show the circles centered at U with radii $\frac{\sqrt{6}-\sqrt{2}}{4}$ and $\frac{\sqrt{6}+\sqrt{2}}{4}$ intersect circle ω, so that we can define I and E to be two of the points of intersection. The minimum distance from U to a point on ω is $\frac{\sqrt{2}-1}{2}$, the maximum distance is $\frac{\sqrt{2}+1}{2}$, and

$$\frac{\sqrt{2}-1}{2} < \frac{\sqrt{6}-\sqrt{2}}{4} < \frac{\sqrt{6}+\sqrt{2}}{4} < \frac{\sqrt{2}+1}{2}.$$

So the circles centered at U with radii $\frac{\sqrt{6}-\sqrt{2}}{4}$ and $\frac{\sqrt{6}+\sqrt{2}}{4}$ must intersect ω and it is indeed possible for $[PIE] = \frac{1}{4}$. $\square$

4/1/31. A group of 100 friends stands in a circle. Initially, one person has 2019 mangos, and no one else has mangos. The friends split the mangos according to the following rules:

- *sharing*: to share, a friend passes two mangos to the left and one mango to the right.
- *eating*: the mangos must also be eaten and enjoyed. However, no friend wants to be selfish and eat too many mangos. Every time a person eats a mango, they must also pass another mango to the right.

A person may only *share* if they have at least three mangos, and they may only *eat* if they have at least two mangos. The friends continue sharing and eating, until so many mangos have been eaten that no one is able to share or eat anymore.

Show that there are exactly eight people stuck with mangos, which can no longer be shared or eaten.

Solution

There is a lot of freedom in how the mangos can be passed, and there are a lot of moving parts. It will help us to find something that stays constant (an invariant) no matter how the mangos are shared or eaten.

Number the people in the circle 0 through 99 so that person 0 is the one starting with 2019 mangos and person 1 is to the right of person 0. Let m_i be the number of mangos held by person i, and let m represent the state $(m_0, m_1, \ldots, m_{99})$. We claim the quantity

$$f(m) = \sum_{i=0}^{99} 2^i m_i \pmod{2^{100} - 1}$$

is invariant under the operations of sharing and eating. When person i shares, m_{i-1} increases by 2, m_i decreases by 3, and m_{i+1} increases by 1 (with $m_{100} = m_0$ for convenience, since $2^{100} \equiv 1 \bmod 2^{100} - 1$). Because

$$2^{i-1}(m_{i-1} + 2) + 2^i(m_i - 3) + 2^{i+1}(m_{i+1} + 1) = 2^{i-1}m_{i-1} + 2^i m_i + 2^{i+1}m_{i+1},$$

$f(m)$ stays constant under sharing. When person i eats, m_i decreases by 2 and m_{i+1} increases by 1.

$$2^i(m_i - 2) + 2^{i+1}(m_{i+1} + 1) = 2^i m_i + 2^{i+1}m_{i+1},$$

so $f(m)$ is constant under eating as well.

Once no more eating or sharing is possible, each person has either 0 or 1 mango. $f(m) = 2019$ to begin, so it must be 2019 at the end as well. Since each m_i is 0 or 1 and $f(m) \not\equiv 0 \bmod 2^{100} - 1$, we must have

$$2019 = \sum_{i=0}^{99} 2^i m_i.$$

Therefore m_i is the ith digit from the right in the binary expansion of 2019. $2019 = 11111100011_2$, which has exactly 8 ones. So exactly 8 friends will still have mangos at the end of this process, and in fact we can even do better and identify exactly which 8 friends have mangos. □

5/1/31. Let n be a positive integer. For integers a, b with $0 \le a, b \le n-1$, let $r_n(a,b)$ denote the remainder when ab is divided by n. If S_n denotes the sum of all n^2 remainders $r_n(a,b)$, prove that

$$\frac{1}{2} - \frac{1}{\sqrt{n}} \le \frac{S_n}{n^3} \le \frac{1}{2}.$$

Solution

To compute S_n, we first fix b and compute the sum

$$\sum_{a=0}^{n-1} r_n(a,b) = r_n(0,b) + r_n(1,b) + \cdots + r_n(n-1,b).$$

Let $d = \gcd(b,n)$. For all k,

$$r_n\left(k + \frac{n}{d}, b\right) = r_n(k,b)$$

because $\left(k + \frac{n}{d}\right) b - kb = n \cdot \frac{b}{d}$, which is a multiple of n. Therefore the sum repeats itself every $\frac{n}{d}$ terms, so it suffices to instead compute the sum

$$r_n(0,b) + r_n(1,b) + \cdots + r_n\left(\frac{n}{d} - 1, b\right)$$

and multiply the result by d. Each of the terms $r_n(a,b)$ in this sum is a multiple of d, because both ab and n are multiples of d. We claim that all the terms of this sum are pairwise distinct: indeed, if $1 \le a_1, a_2 \le \frac{n}{d}$ and $r_n(a_1,b) = r_n(a_2,b)$, then $a_1 b - a_2 b = (a_1 - a_2)b$ is divisible by n. So $a_1 - a_2$ is divisible by $\frac{n}{d}$, which is only possible if $a_1 = a_2$. Therefore the terms in this sum must consist of all the multiples of d from 0 to n (excluding n) exactly once, and

$$\begin{aligned} r_n(0,b) + r_n(1,b) + \cdots + r_n\left(\frac{n}{d} - 1, b\right) &= 0 + d + 2d + \cdots + \left(\frac{n}{d} - 1\right) d \\ &= d\left(0 + 1 + 2 + \cdots + \left(\frac{n}{d} - 1\right)\right) \\ &= d \cdot \frac{1}{2}\left(\frac{n}{d} - 1\right)\left(\frac{n}{d}\right) \\ &= \frac{n(n-d)}{2d} \end{aligned}$$

and so

$$\sum_{a=0}^{n-1} r_n(a,b) = d \cdot \frac{n(n-d)}{2d} = \frac{n(n-d)}{2}.$$

Since $d = \gcd(b, n)$, we have

$$\begin{aligned} S_n &= \sum_{b=0}^{n-1} \sum_{a=0}^{n-1} r_n(a, b) \\ &= \sum_{b=0}^{n-1} \frac{n(n - \gcd(b, n))}{2} \\ &= \frac{n^3}{2} - \frac{n}{2} \sum_{b=0}^{n-1} \gcd(b, n) \end{aligned}$$

and so

$$\frac{S_n}{n^3} = \frac{1}{2} - \frac{1}{2n^2} \sum_{b=0}^{n-1} \gcd(b, n).$$

It remains to estimate the sum

$$T_n = \sum_{b=0}^{n-1} \gcd(b, n) = \sum_{b=1}^{n} \gcd(b, n).$$

Clearly this sum is positive, which immediately proves $S_n \leq \frac{1}{2}$.

For the lower bound, we rewrite the sum T_n as follows. Fix a positive divisor d of n, and note that $\gcd(b, n) = d$ if and only if $\frac{b}{d}$ is an integer with $\gcd\left(\frac{b}{d}, \frac{n}{d}\right) = 1$. Since $1 \leq b \leq n$, we have $1 \leq \frac{b}{d} \leq \frac{n}{d}$, so the number of values of b with $\gcd(b, n) = d$ is exactly $\phi\left(\frac{n}{d}\right)$, where ϕ is the Euler totient function. It follows that

$$T_n = \sum_{b=1}^{n} \gcd(b, n) = \sum_{d|n} \phi\left(\frac{n}{d}\right) d$$

where the sum runs over all positive divisors of n. Since $\phi(m) \leq m$ for every m, we have

$$T_n \leq \sum_{d|n} \frac{n}{d} \cdot d = n \sum_{d|n} 1 = n\tau(n),$$

where $\tau(n)$ denotes the number of divisors of n. Now, whenever k is a divisor of n greater than $\sqrt{n}$, the corresponding divisor $\frac{n}{k}$ is less than $\sqrt{n}$, so n has at most two divisors for every positive integer less than or equal to $\sqrt{n}$. Thus $\tau(n) \leq 2\sqrt{n}$ and

$$T_n \leq n\tau(n) \leq 2n\sqrt{n}.$$

Finally,

$$\frac{S_n}{n^3} = \frac{1}{2} - \frac{T_n}{2n^2} \geq \frac{1}{2} - \frac{2n\sqrt{n}}{2n^2} = \frac{1}{2} - \frac{1}{\sqrt{n}},$$

which is the requested lower bound. □