USA Mathematical Talent Search
Round 3 Solutions
Year 29 — Academic Year 2017–2018
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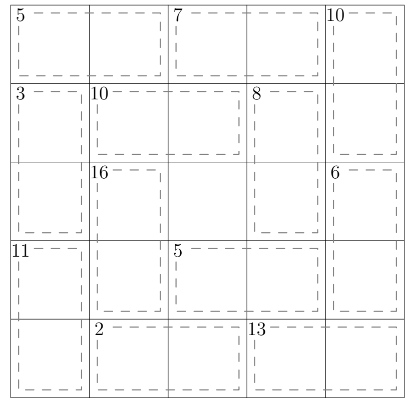
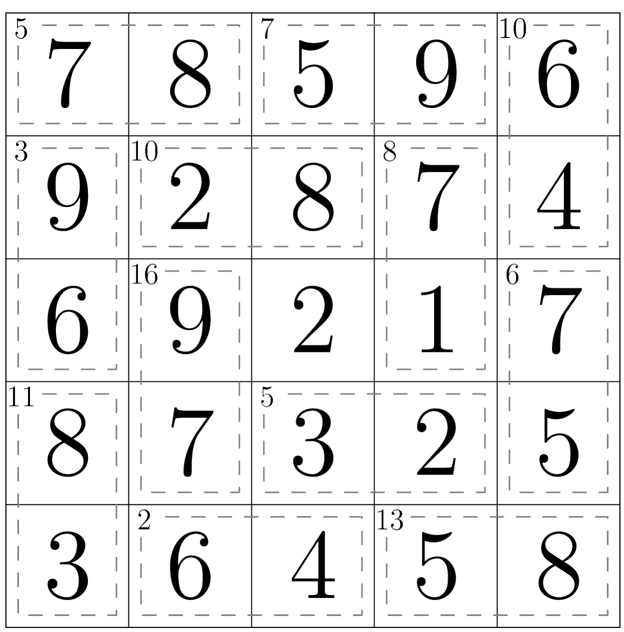
1/3/29. Fill in each cell of the grid with a positive digit so that the following conditions hold:

1. each row and column contains five distinct digits;
2. for any cage containing multiple cells of a row, the label on the cage is the GCD of the sum of the digits in the cage and the sum of the digits in the whole row; and
3. for any cage containing multiple cells of a column, the label on the cage is the GCD of the sum of the digits in the cage and the sum of the digits in the whole column.



You do not need to prove that your answer is the only one possible; you merely need to find an answer that satisfies the constraints above. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

Solution



2/3/29. Let q be a real number. Suppose there are three distinct positive integers a, b, c such that $q + a, q + b, q + c$ is a geometric progression. Show that q is rational.

Solution

Since $q + a$, $q + b$, $q + c$ is a geometric progression,

$$(q + b)^2 = (q + a)(q + c).$$

Expanding, we get $q^2 + 2bq + b^2 = q^2 + (a + c)q + ac$, so

$$(a - 2b + c)q = b^2 - ac.$$

Suppose that $a - 2b + c = 0$. Then we must have $b^2 - ac = 0$, so $a + c = 2b$ and $b^2 = ac$.* Squaring the equation $a + c = 2b$, we get

$$a^2 + 2ac + c^2 = 4b^2,$$

so $a^2 + 2ac + c^2 = 4ac$, which implies $a^2 - 2ac + c^2 = 0$. We can factor this as $(a - c)^2 = 0$, so $a = c$. This is a contradiction, because we are told that a and c are distinct, so $a - 2b + c$ cannot be 0.

We can then safely divide both sides of $(a - 2b + c)q = b^2 - ac$ by $a - 2b + c$ to get

$$q = \frac{b^2 - ac}{a - 2b + c}.$$

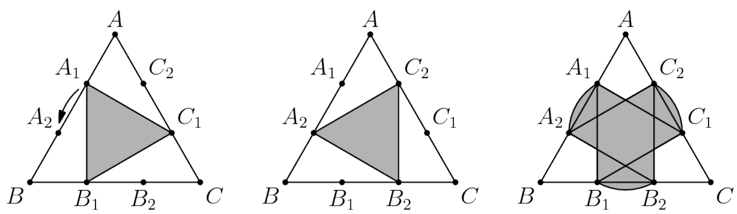
Since a, b, and c are integers, q is rational.

* You may notice that this means that b is both the arithmetic mean (AM) and geometric mean (GM) of a and c. By the famous AM-GM inequality, this means that $a = c$.

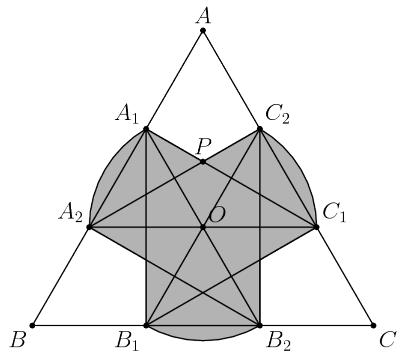
3/3/29. Let ABC be an equilateral triangle with side length 1. Let A_1 and A_2 be the trisection points of AB with A_1 closer to A, B_1 and B_2 be the trisection points of BC with B_1 closer to B, and C_1 and C_2 be the trisection points of CA with C_1 closer to C. Grogg has an orange equilateral triangle the size of triangle $A_1B_1C_1$. He puts the orange triangle over triangle $A_1B_1C_1$ and then rotates it about its center in the shortest direction until its vertices are over $A_2B_2C_2$. Find the area of the region that the orange triangle traveled over during its rotation.

Solution

The diagrams below show the initial position of the triangle, the final position of the triangle, and the region swept by the triangle, respectively. Note that the vertices of the triangle trace arcs of circles.



Let O be the center of equilateral triangle ABC.



By symmetry, triangles OA_1A_2, OA_2B_1, OB_1B_2, OB_2C_1, OC_1C_2, and OC_2A_1 are all equilateral, and they are all congruent with side length $A_1A_2 = 1/3$.

Note that $OA_1C_2C_1$ is a rhombus, so A_1C_1 is perpendicular to OC_2. Similarly, A_1O is perpendicular to A_2C_2. This means that the intersection of $\overline{A_1C_1}$ and $\overline{A_2C_2}$, namely P, is the center of equilateral triangle OA_1C_2. So 1/3 of triangle OA_1C_2 is not included in the region. The same holds for triangles OA_2B_1 and OB_2C_1.

Additionally, OA_1A_2 is a circular sector with a radius of 1/3 and a central angle of 60°. The same holds for sectors OB_1B_2 and OC_1C_2. So, in total the region consists of two-thirds of three equilateral triangles with side length $\frac{1}{3}$ and three circular sectors with radius 1/3 and central angle 60°. Therefore, the area of the region is

$$3 \cdot \frac{2}{3} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{3}\right)^2 + 3 \cdot \frac{1}{6} \cdot \pi \cdot \left(\frac{1}{3}\right)^2 = \boxed{\frac{\sqrt{3}+\pi}{18}}.$$

4/3/29. A positive integer is called *uphill* if the digits in its decimal representation form an increasing sequence from left to right. That is, a number $\overline{a_1 a_2 \cdots a_n}$ is uphill if $a_i \le a_{i+1}$ for all i. For example, 123 and 114 are both uphill. Suppose a polynomial $P(x)$ with rational coefficients takes on an integer value for each uphill positive integer x. Is it necessarily true that $P(x)$ takes on an integer value for each integer x?

Solution (Problem proposed by Nikolai Beluhov)

We claim that it is $\boxed{\text{not true}}$ that $P(x)$ necessarily takes on an integer value for each integer x. To show this, we will construct a polynomial with rational coefficients that takes on an integer value for each uphill positive integer, but does not take on an integer value for every integer input x.

Consider the sequence $b_0 = 1$, $b_1 = 11$, $b_2 = 111, \ldots$. Every uphill integer can be written as a sum of at most nine of the b_i (possibly with repeats). These numbers inspire us to think modulo 11. If i is even, then $b_i \equiv 1 \pmod{11}$. If i is odd, then $b_i \equiv 0 \pmod{11}$. This means that adding nine of these numbers together, we can *never* get a number that is congruent to 10 modulo 11. Hence, no uphill integer is congruent to 10 modulo 11.

So, if we can construct a polynomial that gives an integer exactly when the input is not 10 modulo 11, we will be done. To that end, we think of Fermat's Little Theorem. We know that $(x-10)^{10} \equiv 1 \pmod{11}$ *unless* x is equivalent to 10 modulo 11. So, the polynomial

$$P(x) = \frac{(x-10)^{10} - 1}{11}$$

takes on an integer value for any integer input that is not 10 modulo 11. However, it is never an integer when the input is equivalent to 10 modulo 11, because the numerator is not divisible by 11. Therefore, $P(x)$ takes on an integer value for each uphill positive integer, but does not take on an integer value for every integer input x, as desired.

Note from proposer: A positive integer is called *downhill* if the digits in its decimal representation form a nonstrictly decreasing sequence from left to right. Suppose that a polynomial $P(x)$ with rational coefficients takes on an integer value for each downhill positive integer x. Is it necessarily true that $P(x)$ takes on an integer value for each integer x?

Surprisingly, the downhill problem is a lot more difficult than the uphill one. Here is a sketch of one of its solutions. First off, prove that if n is large enough then there exists some remainder r modulo 2^n such that no downhill positive integer is congruent to r modulo 2^n. Then consider the rational-coefficient polynomial

$$P(x) = \frac{(x-r+1)(x-r+2)\cdots(x-r+2^n-1)}{2\cdot(2^n-1)!}.$$

It takes on an integer value for each downhill positive integer x, and yet $P(r) = \frac{1}{2}$.

5/3/29. Let n be a positive integer. Aavid has a card deck consisting of $2n$ cards, each colored with one of n colors such that every color is on exactly two of the cards. The $2n$ cards are randomly ordered in a stack. Every second, he removes the top card from the stack and places the card into an area called the pit. If the other card of that color also happens to be in the pit, Aavid collects both cards of that color and discards them from the pit.

Of the $(2n)!$ possible original orderings of the deck, determine how many have the following property: at every point, the pit contains cards of at most two distinct colors.

Solution

First, we'll assume that the order of the appearance of the first copy of each color is $1, 2, \ldots, n$ (and hence we'll multiply by $n!$ at the end). That is, we assume that color 1 is the first color to appear, then color 2 is the second unique color we see, color 3 is the third unique color, etc. We will also assume that the two cards of the same color are identical (and hence we'll multiply by 2^n at the end).

Suppose the stream of cards is $s_1, s_2, \ldots, s_{2n}$. After each card s_j is placed:

- If j is odd, there is exactly one card in the pit.
- If j is even, there are either zero or two cards in the pit.

Suppose we're at the odd position s_{2k-1} where $k < n$. Then, we have three ways to deal out the next two cards to get to s_{2k+1}. To see this, suppose that ℓ is the color of the lone card in the pit at s_{2k-1}, and m is the smallest unused color. Then, we can deal out the next two cards as $(\ell, m), (m, \ell)$, or (m, m). For example, the sequence must start in one of these three patterns:

$$\begin{array}{ccc} 1, & 1, & 2 \\ 1, & 2, & 1 \\ 1, & 2, & 2 \end{array}$$

After s_{2n-1} there is only one way to deal out the last card. Therefore, since we have 3 choices for what to deal at each of $s_1, s_3, \ldots, s_{2n-3}$, we have 3^{n-1} ways to deal out the cards. Accounting for our earlier simplifications, we have a total of $\boxed{2^n \cdot n! \cdot 3^{n-1}}$ ways to order the deck.