USA Mathematical Talent Search

Solutions to Problem 1/4/19

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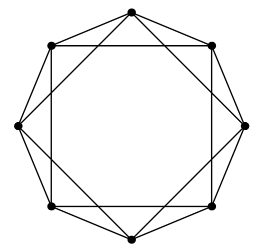
1/4/19. In the diagram at right, each vertex is labeled with a different positive factor of 2008, such that if two vertices are connected by an edge, then the label of one vertex divides the label of the other vertex. In how many different ways can the vertices be labeled? Two labelings are considered the same if one labeling can be obtained by rotating and/or reflecting the other labeling.

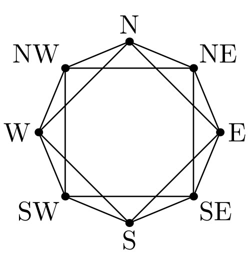


Comments We can solve the problem by using casework. The number of cases can be reduced by using the symmetry in the diagram. Starting with the factors 8 and 251 is also a good idea, because they are the "least connectable" factors. *Solutions edited by Naoki Sato.*

Solution by: Sam Elder (12/CO)

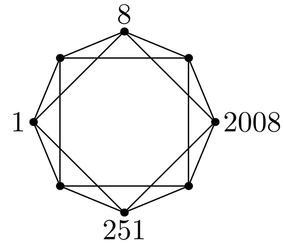
The positive factors of 2008 are 1, 2, 4, 8, 251, 502, 1004, and 2008. Six pairs of these numbers do not satisfy the given property (one divides the other): $(2, 251)$, $(4, 251)$, $(4, 502)$, $(8, 251)$, $(8, 502)$, and $(8, 1004)$. These pairs cannot be connected by an edge.

Denote the vertices as N, NW, W, SW, S, SE, E, and NE as directions on a compass.

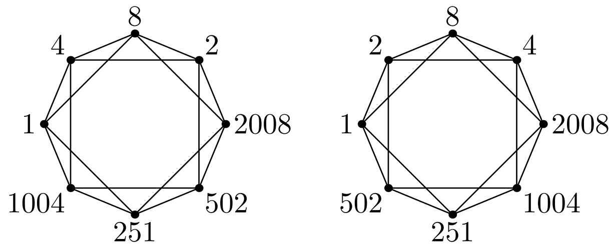


Rotate any working grid so 251 is at S. Each vertex in this diagram is connected by edges to four others, and not connected to the other three. Since 251 cannot be connected to 2, 4, or 8, these three numbers must be in the northernmost three places in the diagram, N, NE and NW. Consider the different positions 8 can be in.

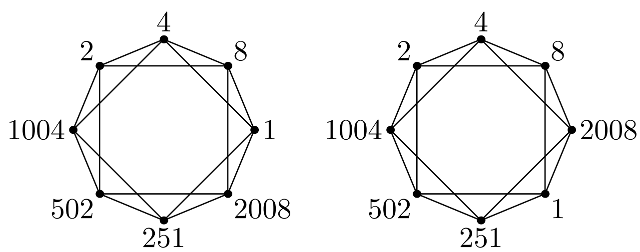
Case 1: 8 is at N. The only two factors that can lie at W and E are 1 and 2008. Reflect any diagrams across the north-south axis so 1 is at W and 2008 is at E.



2 and 4 occupy NE and NW, so 502 and 1004 occupy SE and SW. The only other forbidden pair is $(4, 502)$, so if 4 is at NE then 502 is at SW, and if 4 is at NW then 502 is at SE. These are the two solutions for this case:



Case 2: 8 is not at N. Reflect across the north-south axis so 8 is at NE. Then 2 and 4 are at N and NW, while 502 and 1004 are at W and SW to avoid edges with 8. The only way 4 and 502 cannot share an edge is if 4 is at N and 502 at SW. Then 2 is at NW and 1004 at W, and the only factors left to arrange are 1 and 2008, which can go either at E or SE:



With two possible labelings in each case, there are 4 possible labelings in all.

2/4/19. Determine, with proof, the greatest integer n such that

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor < n,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x.

Credit This problem was proposed by Andy Niedermaier.

Comments In an intuitive (but not rigorous) sense, n should leave the maximum remainder when divided by 2, 3, 11, and 13, i.e. n should leave a remainder of 12 when divided by 13, and so on. The following proof rigorously establishes the answer by finding a bound in terms of these remainders. *Solutions edited by Naoki Sato.*

Solution by: Wenyu Cao (9/NJ)

We claim that the greatest integer that satisfies the given inequality is 1715. First, we check that $n = 1715$ satisfies the given inequality:

$$\left\lfloor \frac{1715}{2} \right\rfloor + \left\lfloor \frac{1715}{3} \right\rfloor + \left\lfloor \frac{1715}{11} \right\rfloor + \left\lfloor \frac{1715}{13} \right\rfloor = 857 + 571 + 155 + 131 = 1714 < 1715.$$

Next, we claim that

$$\left\lfloor \frac{x}{k} \right\rfloor \geq \frac{x - k + 1}{k}$$

for all positive integers x and k. By the Division Algorithm, there exist integers q and r such that $x = qk + r$ and $0 \leq r \leq k - 1$. Then

$$\left\lfloor \frac{x}{k} \right\rfloor = \left\lfloor \frac{qk + r}{k} \right\rfloor = \left\lfloor q + \frac{r}{k} \right\rfloor = q,$$

since $0 \leq r/k < 1$, and

$$\frac{x - k + 1}{k} = \frac{qk + r - k + 1}{k} = q + \frac{r - (k - 1)}{k} \leq q = \left\lfloor \frac{x}{k} \right\rfloor,$$

as desired.

Now, let n be a positive integer that satisfies the given inequality:

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor < n.$$

Since both sides of the inequality are integers,

$$\left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor \leq n - 1.$$

Therefore, from the result above,

$$\begin{aligned} n-1 &\geq \left\lfloor \frac{n}{2} \right\rfloor + \left\lfloor \frac{n}{3} \right\rfloor + \left\lfloor \frac{n}{11} \right\rfloor + \left\lfloor \frac{n}{13} \right\rfloor \\ &\geq \frac{n-1}{2} + \frac{n-2}{3} + \frac{n-10}{11} + \frac{n-12}{13} \\ &= \frac{859}{858}n - \frac{2573}{858}. \end{aligned}$$

Then

$$\frac{n}{858} \leq \frac{1715}{858},$$

so $n \leq 1715$. Since we have shown that $n = 1715$ works, we are done.

3/4/19. Let $0 < \mu < 1$. Define a sequence $\{a_n\}$ of real numbers by $a_1 = 1$ and for all integers $k \geq 1$,

$$a_{2k} = \mu a_k,$$
$$a_{2k+1} = (1-\mu)a_k.$$

Find the value of the sum $\displaystyle\sum_{k=1}^{\infty} a_{2k}a_{2k+1}$ in terms of μ.

Credit This problem was proposed by Sandor Lehoczky, and modified by Dave Patrick.

Comments This following solution deftly finds the required sum by directly using the given recursion relations. *Solutions edited by Naoki Sato.*

Solution by: Tony Jin (10/CA)

By the definition of $\{a_n\}$,

$$\sum_{k=1}^{\infty} a_{2k}a_{2k+1} = \sum_{k=1}^{\infty}[\mu a_k \cdot (1-\mu)a_k] = \mu(1-\mu)\sum_{k=1}^{\infty} a_k^2.$$

We can split up the sum $\sum_{k=1}^{\infty} a_k^2$ as follows:

$$\begin{aligned}
\sum_{k=1}^{\infty} a_k^2 &= a_1^2 + \sum_{k=1}^{\infty} a_{2k}^2 + \sum_{k=1}^{\infty} a_{2k+1}^2 \\
&= a_1^2 + \sum_{k=1}^{\infty} \mu^2 a_k^2 + \sum_{k=1}^{\infty}(1-\mu)^2 a_k^2 \\
&= a_1^2 + \mu^2 \sum_{k=1}^{\infty} a_k^2 + (1-\mu)^2 \sum_{k=1}^{\infty} a_k^2 \\
&= a_1^2 + [\mu^2 + (1-\mu)^2] \sum_{k=1}^{\infty} a_k^2.
\end{aligned}$$

Therefore,

$$[1 - \mu^2 - (1-\mu)^2] \sum_{k=1}^{\infty} a_k^2 = a_1^2,$$

so

$$\sum_{k=1}^{\infty} a_k^2 = \frac{a_1^2}{1-\mu^2-(1-\mu)^2} = \frac{1}{2\mu(1-\mu)}.$$

Finally, the sum we seek is

$$\sum_{k=1}^{\infty} a_{2k}a_{2k+1} = \mu(1-\mu)\sum_{k=1}^{\infty} a_k^2 = \frac{1}{2}.$$

4/4/19. Suppose that w, x, y, z are positive real numbers such that $w + x < y + z$. Prove that it is impossible to simultaneously satisfy both

$$(w + x)yz < wx(y + z) \qquad \text{and} \qquad (w + x)(y + z) < wx + yz.$$

Comments Since we want to show that not all three inequalities can hold simultaneously, we can approach the problem by using contradiction. *Solutions edited by Naoki Sato.*

Solution 1 by: Andy Zhu (11/NJ)

For the sake of contradiction, suppose that all the given inequalities hold. Multiplying the inequalities $wx(y + z) > (w + x)yz$ and $wx + yz > (w + x)(y + z)$, we get

$$wx(y + z)(wx + yz) > yz(w + x)^2(y + z).$$

By the AM-GM inequality, $(w + x)^2 \geq 4wx$, so

$$wx(y + z)(wx + yz) > yz(w + x)^2(y + z) \geq 4wxyz(y + z).$$

Dividing by $wx(y + z)$ (which is positive), we get

$$wx + yz > 4yz,$$

so $wx > 3yz$.

Also, since $y + z > w + x$ and $wx + yz > (w + x)(y + z)$,

$$wx + yz > (w + x)(y + z) > (w + x)^2 \geq 4wx,$$

so $yz > 3wx$. Multiplying the inequalities $wx > 3yz$ and $yz > 3wx$, we get $wxyz > 9wxyz$, contradiction. Thus, not all the given inequalities can hold simultaneously.

Solution 2 by: Kristin Cordwell (11/NM)

We argue by contradiction. Suppose that the positive real numbers w, x, y, z satisfy all the given inequalities, so $w + x < y + z$,

$$(w+x)yz < wx(y+z) \quad \Rightarrow \quad wxy + wxz - wyz - xyz > 0,$$

and

$$(w+x)(y+z) < wx + yz \quad \Rightarrow \quad wx + yz - wy - xy - wz - xz > 0.$$

Now consider the polynomial $p(s) = (s-w)(s-x)(s+y)(s+z)$. Expanding this, we have

$$\begin{aligned} p(s) = s^4 &+ (y+z-w-x)s^3 + (wx+yz-wy-xy-wz-xz)s^2 \\ &+ (wxy+wxz-wyz-xyz)s + wxyz. \end{aligned}$$

The coefficients of s^3, s^2, and s are all positive, and $wxyz > 0$ because w, x, y, $z > 0$. Therefore, $p(s) > 0$ for all $s > 0$. However, $p(w) = 0$ and $w > 0$, contradiction. Therefore, all three inequalities cannot simultaneously hold.

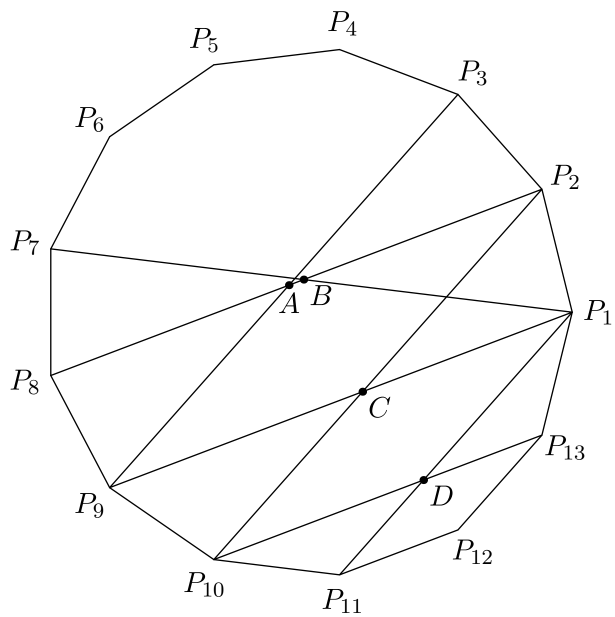
5/4/19. Let $P_1P_2P_3\cdots P_{13}$ be a regular 13-gon. For $1 \le i \le 6$, let $d_i = P_1P_{i+1}$. The 13 diagonals of length d_6 enclose a smaller regular 13-gon, whose side length we denote by s. Express s in the form

$$s = c_1d_1 + c_2d_2 + c_3d_3 + c_4d_4 + c_5d_5 + c_6d_6,$$

where c_1, c_2, c_3, c_4, c_5, and c_6 are integers.

Comments In geometry, when trying to find relationships between lengths, it is often useful to find line segments that add up to the lengths in question. The following solution constructs these line segments by using the symmetry of the regular 13-gon. *Solutions edited by Naoki Sato.*

Solution by: Rui Jin (11/CA)



Let A be the intersection of P_2P_8 and P_3P_9, B the intersection of P_1P_7 and P_2P_8, C the intersection of P_1P_9 and P_2P_{10}, and D the intersection of P_1P_{11} and $P_{10}P_{13}$. Since P_1P_7, P_2P_8, and P_3P_9 are all diagonals of length d_6, AB is a side of the smaller regular 13-gon, so $s = AB$.

By symmetry, $BP_2 = AP_8$. Let $x = BP_2 = AP_8$. Since $P_2P_8 = P_1P_7 = d_6$, $AP_2 = d_6 - x$.

Diagonals P_3P_9 and P_2P_{10} are parallel, and diagonals P_2P_8 and P_1P_9 are parallel, so quadrilateral P_2AP_9C is a parallelogram. Hence, $AP_2 = d_6 - x = P_9C$. Since $P_1P_9 = P_1P_6 = d_5$, $P_9C = d_6 - x = d_5 - P_1C$.

Since P_2P_{10} and P_1P_{11} are parallel and P_1P_9 and $P_{13}P_{10}$ are parallel, quadrilateral $P_1CP_{10}D$ is a parallelogram. Hence, $P_1C = DP_{10}$ and we can rewrite the equation above as $d_6 - x = d_5 - DP_{10}$. Since $P_{10}P_{13} = P_1P_4 = d_3$, $d_6 - x = d_5 - (d_3 - DP_{13})$.

Finally, since P_1P_{11} and $P_{13}P_{12}$ are parallel and $P_{13}P_{10}$ and $P_{12}P_{11}$ are parallel, quadrilateral $P_{13}DP_{11}P_{12}$ is a parallelogram. Hence, $DP_{13} = P_{11}P_{12}$ and we can rewrite the equation above as $d_6 - x = d_5 - (d_3 - P_{11}P_{12})$. Since $P_{11}P_{12} = P_1P_2 = d_1$, $d_6 - x = d_5 - (d_3 - d_1)$.

Rearranging the last equation, we find that $x = d_6 - d_5 + d_3 - d_1$. Since $s = d_6 - 2x$, we have

$$s = d_6 - 2(d_6 - d_5 + d_3 - d_1) = 2d_1 - 2d_3 + 2d_5 - d_6.$$