1/3/19. We construct a sculpture consisting of infinitely many cubes, as follows. Start with a cube with side length 1. Then, at the center of each face, attach a cube with side length $\frac{1}{3}$ (so that the center of a face of each attached cube is the center of a face of the original cube). Continue this procedure indefinitely: at the center of each exposed face of a cube in the structure, attach (in the same fashion) a smaller cube with side length one-third that of the exposed face. What is the volume of the entire sculpture?

Comments Once the geometry of the sculpture has been determined, the volume can be found by summing an infinite geometric sequence. *Solutions edited by Naoki Sato.*

Solution by: Dmitri Gekhtman (11/IN)

Since the cube with side length 1 has 6 faces, the sculpture has 6 cubes of side length $\frac{1}{3}$ and volume $(\frac{1}{3})^3 = \frac{1}{27}$. Since each of the 6 cubes of side length $\frac{1}{3}$ has 5 exposed faces, there are 6×5 cubes of side length $(\frac{1}{3})^2$ and volume $(\frac{1}{27})^2$. By the same reasoning, there are 6×5^2 cubes of volume $(\frac{1}{27})^n$. In general, the sculpture contains $6 \times 5^{n-1}$ cubes of volume $(\frac{1}{27})^n$, where n is a positive integer. The sculpture contains one cube of volume 1. Therefore, the total volume of the sculpture is

$$1 + \sum_{n=1}^{\infty} 6 \times 5^{n-1} \times \left(\frac{1}{27}\right)^n = 1 + \sum_{n=0}^{\infty} \frac{6}{27} \times \left(\frac{5}{27}\right)^n.$$

Since the infinite sum is an infinite geometric series with first term $\frac{6}{27}$ and common ratio $\frac{5}{27}$, the volume is

$$1 + \frac{\frac{6}{27}}{1 - \frac{5}{27}} = \frac{14}{11}.$$

2/3/19. Gene starts with the 3×3 grid of 0's shown at left below. He then repeatedly chooses a 2×2 square within the grid and increases all four numbers in the chosen 2×2 square by 1. One possibility for Gene's first three steps is shown below.

0	0	0
0	0	0
0	0	0

$\rightarrow$

0	0	0
1	1	0
1	1	0

$\rightarrow$

0	1	1
1	2	1
1	1	0

$\rightarrow$

0	1	1
2	3	1
2	2	0

How many different grids can be produced with this method such that each box contains an integer from 1 to 12, inclusive? (The numbers in the boxes need not be distinct.)

Comments The number of different grids can be counted by identifying each grid with a quintuple of positive integers. Then the number of such quintuples can be found using a standard partition argument. *Solutions edited by Naoki Sato.*

Solution by: Matt Superdock (11/PA)

Any 2×2 square inside the 3×3 grid includes the center box, so the integer in the center box is equal to the number of steps Gene makes. Therefore, Gene can make at most 12 steps, or else the middle box will contain an integer greater than 12. Additionally, each corner box of the 3×3 grid is included in only one of the four 2×2 squares. Therefore, Gene must choose each 2×2 square at least once, or else one of the corner boxes will contain a 0. The final grid depends only on how many times Gene chooses each 2×2 square, not which order he chooses them.

Let a, b, c, and d be the numbers of times Gene chooses each 2×2 square. Then a, b, c, and d are positive integers, and $a + b + c + d \leq 12$. Let $e = 13 - a - b - c - d$, so that e is also a positive integer, and $a + b + c + d + e = 13$. To find the number of solutions to this equation, we consider partitioning 13 objects into 5 non-empty groups. We arrange the 13 objects in a row, and we partition them by placing dividers in 4 of the 12 spaces between adjacent objects. There are $\binom{12}{4} = 495$ ways to do this, so there are 495 grids that can be produced.

3/3/19. Consider all polynomials $f(x)$ with integer coefficients such that $f(200) = f(7) = 2007$ and $0 < f(0) < 2007$. Show that the value of $f(0)$ does not depend on the choice of polynomial, and find $f(0)$.

Comments This problem may be solved by using the following crucial result: If $p(x)$ is a polynomial with integer coefficients, then for any integers a and b, $p(a) - p(b)$ is a multiple of $a - b$. In particular, for the case $b = 0$, $p(a) - p(0)$ is a multiple of a. *Solutions edited by Naoki Sato.*

Solution by: Andy Zhu (11/NJ)

Since all polynomials $P(x)$ with integer coefficients can be expressed in the form $P(x) = x \cdot Q(x) + P(0)$, where $Q(x)$ is a polynomial with integer coefficients, $P(n) \equiv P(0) \pmod{n}$ for all positive integers n. Thus $f(0) \equiv f(7) \equiv 2007 \pmod{7}$ and $f(0) \equiv f(200) \equiv 2007 \pmod{200}$.

Since $\gcd\{7, 200\} = 1$, we can apply the Chinese Remainder Theorem to get $f(0) \equiv 2007 \equiv 607 \pmod{1400}$. The unique value which $f(0)$ takes in the range $0 < f(0) < 2007$ is 607.

4/3/19. Prove that 101 divides infinitely many of the numbers in the set

$$\{2007, 20072007, 200720072007, 2007200720072007, \ldots\}.$$

Comments There are several possible approaches, but the following solution elegantly reduces the elements of the set modulo 101 to a simple formula. *Solutions edited by Naoki Sato.*

Solution by: Daniel Tsai (12/NJ)

Modulo 101, we have $10^0 \equiv 1$, $10^1 \equiv 10$, $10^2 \equiv 100$, $10^3 \equiv 91$, and $10^4 \equiv 1$. Therefore, modulo 101, for each integer $k \geq 0$, $10^{4k} \equiv 1$, $10^{4k+1} \equiv 10$, $10^{4k+2} \equiv 100$, and $10^{4k+3} \equiv 91$. Consequently, for each positive integer n,

$$20072007 \cdots 2007 \equiv (2 \cdot 91 + 0 \cdot 100 + 0 \cdot 10 + 7 \cdot 1)n \equiv 189n \equiv 88n \pmod{101},$$

where the left-hand-side consists of n 2007s. For each positive multiple n of 101, $88n \equiv 0 \pmod{101}$, thus 101 divides infinitely many numbers in the set

$$\{2007, 20072007, 200720072007, 2007200720072007, \ldots\}.$$

5/3/19. For every rational number $0 < \dfrac{p}{q} < 1$, where p and q are relatively prime, construct a circle with center $\left(\dfrac{p}{q}, \dfrac{1}{2q^2}\right)$ and diameter $\dfrac{1}{q^2}$. Also construct circles centered at $\left(0, \dfrac{1}{2}\right)$ and $\left(1, \dfrac{1}{2}\right)$ with diameter 1.
(a) Prove that any two such circles intersect in at most 1 point.
(b) Prove that the total area of all of the circles is $\dfrac{\pi}{4}\left(1 + \dfrac{\sum_{i=1}^{\infty} \frac{1}{i^3}}{\sum_{i=1}^{\infty} \frac{1}{i^4}}\right)$.

Comments Part (a) may be solved by comparing the distance between the centers of two circles to the sum of the radii. Part (b) may be solved by expressing the same sum two different ways, one involving the circles in the problem, and the other involving the infinite sums $\sum_{i=1}^{\infty} \frac{1}{i^3}$ and $\sum_{i=1}^{\infty} \frac{1}{i^4}$. *Solutions edited by Naoki Sato.*

Solution by: Dmitri Gekhtman (11/IN)

(a) Let r and s be two nonnegative integers such that $0 \le r \le s$ and either r and s are relatively prime or $s = 1$. For all possible values of r and s, construct a circle with center $(\frac{r}{s}, \frac{1}{2s^2})$ and diameter $\frac{1}{s^2}$. The resulting set of circles is the same as the one described in the problem. (The parameters $r = 0$, $s = 1$ and $r = 1$, $s = 1$ specify the circles centered at $(0, \frac{1}{2})$ and $(1, \frac{1}{2})$ with diameter 1, respectively.)

Choose two distinct circles from the set of constructed circles. Let one have center $(\frac{r_1}{s_1}, \frac{1}{2s_1^2})$ and diameter $\frac{1}{s_1^2}$ and the other have center $(\frac{r_2}{s_2}, \frac{1}{2s_2^2})$ and diameter $\frac{1}{s_2^2}$. Since, for each circle, the radius is equal to the y-coordinate of the center, both circles are tangent to the x-axis and lie above it. Therefore, one circle cannot lie in the interior of the other circle. The x-coordinate of the center of each circle is the same as the x-coordinate of the intersection of the circle and the x-axis. Since the circles are tangent to the the x-axis at different points, they obviously cannot be internally tangent. Therefore, either (1) the two circles are external to each other and do not intersect at any point, (2) the two circles are externally tangent to each other and intersect at exactly one point, or (3) the two circles intersect at exactly two points.

Let d be the distance between the centers of the two circles, and let R be the sum of the their radii. In case (1), $d > R$. In case (2), $d = R$. In case (3), $d < R$. The distance between the centers of the two circles is

$$d = \sqrt{\left(\frac{r_1}{s_1} - \frac{r_2}{s_2}\right)^2 + \left(\frac{1}{2s_1^2} - \frac{1}{2s_2^2}\right)^2}.$$

The sum of the radii of the two circles is

$$R = \frac{1}{2s_1^2} + \frac{1}{2s_2^2}.$$

So

$$\begin{aligned} d^2 - R^2 &= \left(\frac{r_1}{s_1} - \frac{r_2}{s_2}\right)^2 + \frac{1}{4}\left[\left(\frac{1}{s_1^2} - \frac{1}{s_2^2}\right)^2 - \left(\frac{1}{s_1^2} + \frac{1}{s_2^2}\right)^2\right] \\ &= \frac{r_1^2}{s_1^2} - \frac{2r_1r_2}{s_1s_2} + \frac{r_2^2}{s_2^2} - \frac{1}{s_1^2s_2^2} \\ &= \frac{r_1^2s_2^2 - 2r_1s_2r_2s_1 + r_2^2s_1^2 - 1}{s_1^2s_2^2} \\ &= \frac{(r_1s_2 - r_2s_1)^2 - 1}{s_1^2s_2^2}. \end{aligned}$$

Since r_1, r_2, s_1, and s_2 are integers, $r_1s_2 - r_2s_1$ is an integer. Suppose that $r_1s_2 - r_2s_1 = 0$. Then

$$\frac{r_1}{s_1} = \frac{r_2}{s_2}.$$

But $\frac{r_1}{s_1}$ and $\frac{r_2}{s_2}$ are distinct rational numbers, so $r_1s_2 - r_2s_1 \neq 0$. Since $r_1s_2 - r_2s_1$ is a nonzero integer, $(r_1s_2 - r_2s_1)^2 \geq 1$, so

$$d^2 - R^2 = \frac{(r_1s_2 - r_2s_1)^2 - 1}{s_1^2s_2^2} \geq 0,$$

which means $d^2 \geq R^2$. Since d and R are positive, $d \geq R$. Therefore, the circles cannot intersect at two points. Furthermore, d can equal R (for example, when $r_1 = 0$, $s_1 = 1$, $r_2 = 1$, and $s_2 = 1$, $d = R = 1$). So any two circles can intersect in at most 1 point. This completes the proof.

(b) Consider all pairs of nonnegative integers a and b such that $0 \leq a \leq b$ and $b \neq 0$. The sum over all such pairs (a, b) of $\frac{1}{b^4}$ is

$$\sum_{b=1}^{\infty}\sum_{a=0}^{b}\frac{1}{b^4} = \sum_{b=1}^{\infty}\frac{b+1}{b^4}.$$

Note that each pair (a, b) can be uniquely written in the form (nr, ns), where n is a positive integer and (r, s) is a pair of integers of the type described in part (a). Thus, we may write the sum over all such pairs (a, b) of $\frac{1}{b^4}$ as the sum over all triples (r, s, n) of $\frac{1}{(ns)^4}$. In other words (note that all series below are absolutely convergent),

$$\sum_{b=1}^{\infty}\frac{b+1}{b^4} = \sum_{(r,s)}\sum_{n=1}^{\infty}\frac{1}{(ns)^4} = \sum_{(r,s)}\frac{1}{s^4}\sum_{n=1}^{\infty}\frac{1}{n^4}.$$

Therefore,

$$\sum_{(r,s)} \frac{1}{s^4} = \frac{\displaystyle\sum_{b=1}^{\infty} \frac{b+1}{b^4}}{\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}} = \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4} + \sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}} = 1 + \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}}.$$

Multiplying both sides of this equation by $\frac{\pi}{4}$, we get

$$\sum_{(r,s)} \frac{\pi}{4s^4} = \frac{\pi}{4}\left(1 + \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}}\right).$$

The left side of the equation is the sum over all such pairs (r, s) of the area of a circle of diameter $\frac{1}{s^2}$. Hence, the total area of all of the circles is

$$\frac{\pi}{4}\left(1 + \frac{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^3}}{\displaystyle\sum_{i=1}^{\infty} \frac{1}{i^4}}\right).$$

This completes the proof.