1/2/19. Find the smallest positive integer n such that every possible coloring of the integers from 1 to n with each integer either red or blue has at least one arithmetic progression of three different integers of the same color.

Comments Any solution to this problem will inevitably require some casework. However, by choosing them carefully, the number of cases can be considerably reduced. *Solutions edited by Naoki Sato.*

Solution by: Adrian Chan (12/CA)

We first prove that $n \leq 8$ does not suffice. To do so, it is sufficient to give a counterexample for $n = 8$: Let 1, 4, 5, and 8 be red, and let 2, 3, 6, and 7 be blue. We see that there are no arithmetic sequences among the red numbers, or the blue numbers.

Now we show that for every coloring of the integers from 1 to 9, there is always an arithmetic sequence of three different integers of the same color. For the sake of contradiction, suppose that there is a coloring that does not produce any such arithmetic sequences. Without loss of generality, let 5 be blue. Then at least one of 1 and 9 must be red, otherwise 1, 5, and 9 will form a blue arithmetic sequence.

Case 1: 1 is blue and 9 is red, or 1 is red and 9 is blue.

First, assume that 1 is blue and 9 is red. Then 3 must be red, otherwise 1, 3, and 5 will form a blue arithmetic sequence. Next, 6 must be blue, otherwise 3, 6, and 9 will form a red arithmetic sequence. Next, both 4 and 7 must be red, otherwise 4, 5, and 6, or 5, 6, and 7 will form a blue arithmetic sequence. Finally, both 2 and 8 must be blue, otherwise 2, 3, and 4, or 7, 8, and 9 will form a red arithmetic sequence. However, we end up with 2, 5, and 8 forming a blue arithemtic sequence, contradiction. The case that 1 is red and 9 is blue can be similarly proven, by reversing the order of the colors.

Case 2: Both 1 and 9 are red.

First, assume that 7 is red. Then both 4 and 8 must be blue, otherwise 1, 4, and 7, or 7, 8, and 9 will form a red arithmetic sequence. Next, both 3 and 6 must be red, otherwise 3, 4, and 5, or 4, 5, 6 will form a blue arithmetic sequence. However, we end up with 3, 6, and 9 forming a red arithmetic sequence, contradiction.

Now, asssume that 7 is blue. Then 3 must be red, otherwise 3, 5, and 7 will form a blue arithmetic sequence. Next, 6 must be blue, otherwise 3, 6, and 9 will form a red arithmetic sequence. However, we end up with 5, 6, and 7 forming a blue arithmetic sequence, contradiction.

We conclude that $n = 9$ is the smallest possible integer that satisfies the desired conditions.

2/2/19. Let x, y, and z be complex numbers such that $x+y+z = x^5+y^5+z^5 = 0$ and $x^3+y^3+z^3 = 3$. Find all possible values of $x^{2007}+y^{2007}+z^{2007}$.

Comments There are different ways to appoach this problem. The solution below uses a substitution to eliminate one of the variables, and determines the values of x^3, y^3, and z^3 directly. *Solutions edited by Naoki Sato.*

Solution by: Kristin Cordwell (11/NM)

First, $x+y+z = 0$, so $x+y = -z$. We then cube both sides to get $x^3+3x^2y+3y^2x+y^3 = -z^3$. We rearrange the equation to get $x^3+y^3+z^3 = -3x^2y-3y^2x$. We know that $x^3+y^3+z^3 = 3$, so we get $-3xy(x+y) = 3$, or $xy(x+y) = -1$. This also tells us that neither xy nor $x+y$ can be equal to 0.

Now we take the fifth power of $x+y = -z$ to get

$$x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5 = -z^5.$$

Rearranging the equation gives us

$$x^5+y^5+z^5 = -5xy(x^3+2x^2y+2xy^2+y^3).$$

We know that $x^5+y^5+z^5 = 0$, so we get $-5xy(x^3+2x^2y+2xy^2+y^3) = 0$, or

$$xy(x^3+2x^2y+2xy^2+y^3) = 0.$$

Also, we know that $xy \neq 0$, so we get

$$x^3+2x^2y+2xy^2+y^3 = x^3+y^3+2xy(x+y) = 0.$$

We know that $xy(x+y) = -1$, so this simplifies as $x^3+y^3 = 2$. Finally, we know that $x^3+y^3+z^3 = 3$, so z^3 must equal 1. By symmetry, $x^3 = y^3 = 1$. Since 2007 is divisible by 3, $x^{2007}+y^{2007}+z^{2007} = 3$.

To show that this value is possible, let $x = 1$, $y = -\frac{1}{2}+\frac{\sqrt{3}}{2}i$ and $z = -\frac{1}{2}-\frac{\sqrt{3}}{2}i$, which are the cube roots of unity. Then $x+y+z = x^5+y^5+z^5 = 0$, and $x^3+y^3+z^3 = x^{2007}+y^{2007}+z^{2007} = 3$.

3/2/19. A triangular array of positive integers is called *remarkable* if all of its entries are distinct, and each entry, other than those in the top row, is the quotient of the two numbers immediately above it. For example, the following triangular array is remarkable:

$$\begin{array}{ccccc} 7 & & 42 & & 14 \\ & 6 & & 3 & \\ & & 2 & & \end{array}$$

Find the smallest positive integer that can occur as the greatest element in a remarkable array with four numbers in the top row.

Comments It is fairly easy to find an example where the greatest number in the array is 120. Then, one can prove that 120 is the mininum by showing that some number in the top row is the product of four different integers, all at least 2. *Solutions edited by Naoki Sato.*

Solution by: Dmitri Gekhtman (11/IN)

A remarkable array cannot contain a 1, because this would mean that it contains at least two equal numbers. Denote the integer in the bottom row by a_1. Then the second row from the bottom contains integers a_2 and a_1a_2, with $a_2 \neq a_1$. In the third row from the bottom, and above a_1a_2, there are integers a_3 and $a_1a_2a_3$ (in either order), with a_3 different from a_1 and a_2. Finally, in the top row, and above $a_1a_2a_3$, there are integers a_4 and $a_1a_2a_3a_4$ (in either order), with a_4 different from a_1, a_2, and a_3.

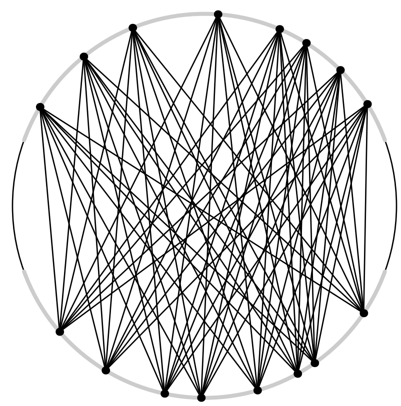
This means that the greatest number in the top row is at least $a_1a_2a_3a_4$. Since $a_1a_2a_3a_4$ is a product of four different integers, all at least 2, it is greater or equal than $2 \cdot 3 \cdot 4 \cdot 5 = 120$. Therefore, the answer to the problem is at least 120. An example with the greatest element in the array equal to 120 is as follows:

$$\begin{array}{ccccccc} 40 & & 5 & & 120 & & 30 \\ & 8 & & 24 & & 4 & \\ & & 3 & & 6 & & \\ & & & 2 & & & \end{array}$$

Hence, the answer is 120.

4/2/19. Two nonoverlapping arcs of a circle are chosen. Eight distinct points are then chosen on each arc. All 64 segments connecting a chosen point on one arc to a chosen point on the other arc are drawn. How many triangles are formed that have at least one of the 16 points as a vertex?

A sample figure is shown below:



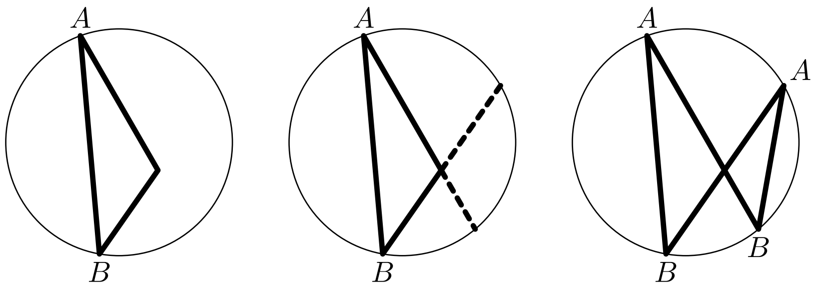
Comments We can count the number of triangles by counting the number of ways to choose points on the circle that give rise to these triangles. However, for each case, we must be careful to factor in just how many triangles are created by the points we choose. *Solutions edited by Naoki Sato.*

Solution by: Santhosh Karnik (11/GA)

To generalize, suppose m points are chosen on one arc and n points are chosen on the other arc. Let the arc with m points be arc A and the arc with n points be arc B. Since all triangles formed must have at least one vertex on the circle, a triangle must have either 1, 2, or 3 vertices on the circle.

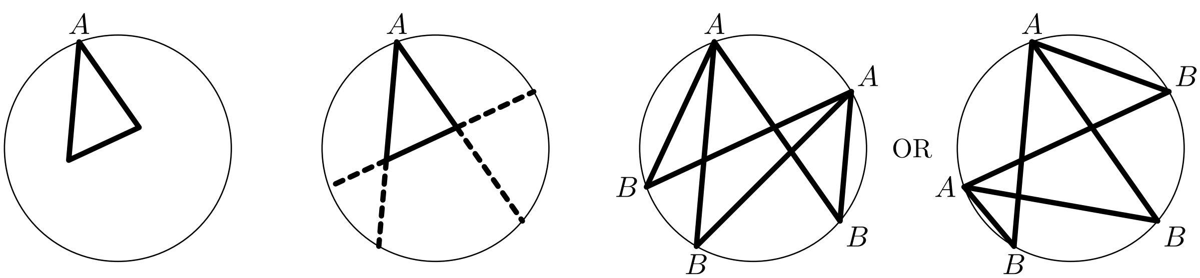
If all 3 vertices of a triangle are on the circle, then at least 2 of the 3 vertices must be on the same arc. But none of the segments connect 2 points on the same arc. Thus there are no triangles formed with all 3 vertices on the circle.

If 2 vertices of a triangle are on the circle, then one must be on the arc A, and the other must be on arc B. The third vertex can be anywhere inside the circle as long as it is connected to the other two vertices. Since all segments connect a point on arc A to a point on arc B, the two segments of the triangle that are connected to the third vertex can each be extended to a point on the circle. This gives a total of 4 points, 2 on each arc. Connecting all possible segments on these 4 points yields 2 triangles with 2 vertices on the circle. Thus, selecting 2 points on arc A and 2 points on arc B always yields 2 triangles with exactly two vertices on the circle, and all such triangles can be formed by choosing 2 points on each arc and connecting all possible segments.



Therefore, there are $2\binom{m}{2}\binom{n}{2}$ triangles formed such that 2 vertices of a triangle are on the circle.

If a triangle has only 1 vertex on the circle, then the other 2 vertices must be inside the circle and connected to each other and the first vertex. By extending all the segments of the triangle to points on the circle, 4 additional points are obtained, 1 on the same arc as the first vertex, and 3 on the other arc. Connecting all possible segments on these 5 points yields 2 triangles with exactly one vertex on the circle. Thus, selecting 2 points on arc A and 3 points on arc B or vice-versa always yields 2 triangles with 1 vertex on the circle, and all such triangles can be formed by choosing 2 points on one arc and 3 points on the other arc and connecting all possible segments.



Therefore there are $2\binom{m}{2}\binom{n}{3} + 2\binom{m}{3}\binom{n}{2}$ triangles formed such that 1 vertex of the triangle is on the circle. Therefore, the total number of triangles formed with at least one vertex on the circle is

$$T(m,n) = 2\left[\binom{m}{2}\binom{n}{2} + \binom{m}{2}\binom{n}{3} + \binom{m}{3}\binom{n}{2}\right].$$

In the problem, there are $m = n = 8$ points on each arc. So the total number of triangles formed with at least one vertex on the circle is

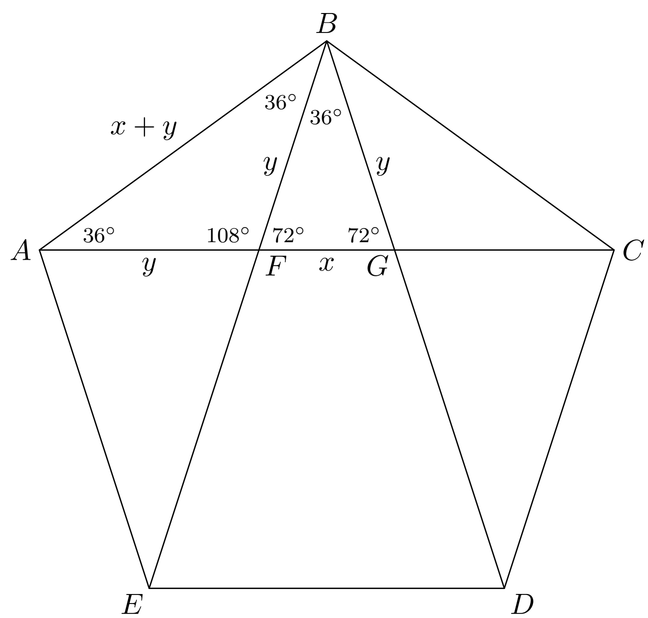
$$T(8,8) = 2\left[\binom{8}{2}\binom{8}{2} + \binom{8}{2}\binom{8}{3} + \binom{8}{3}\binom{8}{2}\right] = 7840.$$

5/2/19. Faces ABC and XYZ of a regular icosahedron are parallel, with the vertices labeled such that $\overline{AX}$, $\overline{BY}$, and $\overline{CZ}$ are concurrent. Let $\mathcal{S}$ be the solid with faces ABC, AYZ, BXZ, CXY, XBC, YAC, ZAB, and XYZ. If $AB = 6$, what is the volume of $\mathcal{S}$?

Comments The volume of $\mathcal{S}$ can be found by relating it to other polyhedra whose volumes are known. In the solution below, the polyhedron $\mathcal{S}$ is related to a hexagonal prism. *Solutions edited by Naoki Sato.*

Solution by: Luyi Zhang (9/CT)

First, we claim that in a regular pentagon, the ratio of the lengths of a diagonal to a side is $\frac{1+\sqrt{5}}{2} : 1$. Let $ABCDE$ be a regular pentagon. Let F be the intersection of $\overline{AC}$ and $\overline{BE}$, and let G be the intersection of $\overline{AC}$ and $\overline{BD}$.



Let $x = FG$ and $y = BG$. Triangle BFG is isosceles, so $BF = BG = y$. Triangle ABF is isosceles, so $AF = BF = y$. Then $AG = x + y$, and since triangle ABG is isosceles, $AB = x + y$.

By AAA, triangles ABG and BFG are similar, so

$$\frac{AB}{BG} = \frac{BF}{FG} \quad \Rightarrow \quad \frac{x+y}{y} = \frac{y}{x},$$

which simplifies as $y^2 - xy - x^2 = 0$. By the quadratic formula,

$$y = \frac{x \pm \sqrt{5x^2}}{2} = \frac{1 \pm \sqrt{5}}{2}\, x.$$

Since $\frac{1-\sqrt{5}}{2}$ is negative,

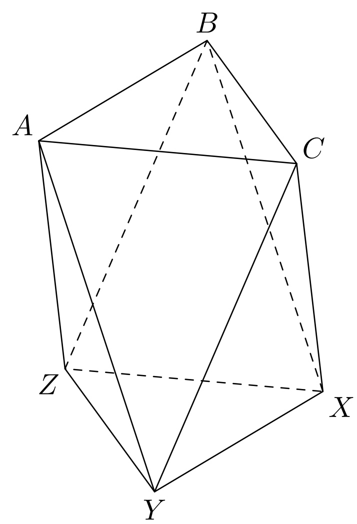
$$y = \frac{1+\sqrt{5}}{2}\,x.$$

By AAA, triangles ABC and AFB are similar, so the ratio of the lengths of a diagonal to a side is

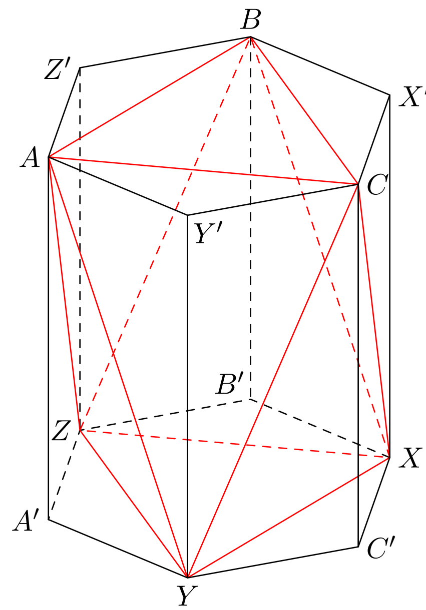
$$\frac{AC}{AB} = \frac{AB}{AF} = \frac{x+y}{y} = \frac{y}{x} = \frac{1+\sqrt{5}}{2},$$

as desired.

Now, the polyhedron $\mathcal{S}$ has eight faces. Two of them are equilateral triangles with the same side length as the icosahedron, which is 6. The other six are isosceles triangles. In these six faces, the base is the same as the side length of the icosahedron, which is 6, and the legs are the diagonals of a regular pentagon with side length 6, which is $3(1+\sqrt{5})$.



Let A', B', and C' be the projections of A, B, and C onto the plane of triangle XYZ, and let X', Y', and Z' be the projections of X, Y, and Z onto the plane of triangle ABC. Then $AY'CX'BZ'A'YC'XB'Z$ is a regular hexagonal prism. (This follows from the symmetry of the icosahedron.)



We see that this hexagonal prism is the union of polyhedron $\mathcal{S}$ (whose edges are in red) and the six tetrahedra $AA'YZ$, $BB'XZ$, $CC'XY$, $XX'BC$, $YY'AC$, and $ZZ'AB$, all of which are congruent. Thus, we can find the volume of $\mathcal{S}$ by finding the volume of the hexagonal prism, and then subtracting the volume of the six tetrahedra.

Let b denote the area of regular hexagon $AY'CX'BZ'$. Since $AB = 6$, the side length of hexagon $AY'CX'BZ'$ is $2\sqrt{3}$. Hence, the hexagon is composed of six equilateral triangles of side length $2\sqrt{3}$, so

$$b = 6 \cdot \frac{\sqrt{3}}{4}(2\sqrt{3})^2 = 18\sqrt{3}.$$

We can then use Pythagoras to find the height h of the prism. Since $\angle AA'Y = 90^\circ$, $AY^2 = (AA')^2 + (A'Y)^2$. But $AY = 3(1+\sqrt{5})$ and $A'Y = 2\sqrt{3}$, so

$$\begin{aligned}(AA')^2 &= AY^2 - (A'Y)^2 \\ &= [3(1+\sqrt{5})]^2 - (2\sqrt{3})^2 \\ &= 42 + 18\sqrt{5}.\end{aligned}$$

Therefore, $h = AA' = \sqrt{42 + 18\sqrt{5}} = \sqrt{3}(3+\sqrt{5})$.

The volume of the hexagonal prism is bh. Since triangle $A'YZ$ has $\frac{1}{6}$th the area of hexagon $AY'CX'BZ'$, the volume of tetrahedron $AA'YZ$ is

$$\frac{1}{3} \cdot \frac{1}{6}b \cdot h = \frac{1}{18}bh.$$

All six tetrahedra $AA'YZ$, $BB'XZ$, $CC'XY$, $XX'BC$, $YY'AC$, $ZZ'AB$ have the same volume, so the volume of $\mathcal{S}$ is

$$bh - \frac{6}{18}bh = bh - \frac{1}{3}bh = \frac{2}{3}bh = \frac{2}{3} \cdot 18\sqrt{3} \cdot \sqrt{3}(3+\sqrt{5}) = 108 + 36\sqrt{5}.$$