**1/1/19.** Gene has $2n$ pieces of paper numbered 1 through $2n$. He removes $n$ pieces of paper that are numbered consecutively. The sum of the numbers on the remaining pieces of paper is 1615. Find all possible values of $n$

**Credit** This problem was proposed by Richard Rusczyk.

**Comments** The first step in the problem is to use algebra to find suitable bounds on $n$. We can then use divisibility properties of integers to find the solutions. *Solutions edited by Naoki Sato.*

**Solution 1 by: Carlos Dominguez (11/OH)**

The minimum sum of the numbers on the remaining pieces is $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$, so $\frac{n(n+1)}{2} \le 1615$. Clearing the denominator and expanding gives $n^2 + n \le 3230$. Since $56^2 + 56 = 3192 < 3230$ and $57^2 + 57 = 3306 > 3230$, we must have $n \le 56$.

The maximum sum of the numbers on the remaining pieces is $(n+1)+(n+2)+\cdots+2n = \frac{n(3n+1)}{2}$, so $\frac{n(3n+1)}{2} \ge 1615$, which implies $3n^2 + n \ge 3230$. Since $3 \cdot 32^2 + 32 = 3104 < 3230$ and $3 \cdot 33^2 + 33 = 3300 > 3230$, we must have $n \ge 33$.

Of the $n$ numbers removed, let $k$ be the first number. Then the sum of the $n$ remaining numbers is

$$\begin{aligned}
&(1 + 2 + \cdots + 2n) - [k + (k+1) + \cdots + (k+n-1)] \\
&= \frac{2n(2n+1)}{2} - \frac{(2k+n-1)n}{2} \\
&= 1615.
\end{aligned}$$

Multiplying both sides by $2/n$ and expanding, we get

$$4n + 2 - (2k + n - 1) = 3n - 2k + 3 = \frac{3230}{n}.$$

Since $3n - 2k + 3$ is an integer, $3230/n$ is also an integer. In other words, $n$ is a factor of 3230. The factors of 3230 are 1, 2, 5, 10, 17, 19, 34, 38, 85, 95, 170, 190, 323, 646, 1615, and 3230. The only factors between 33 and 56 (inclusive) are $n = 34$ and $n = 38$. The corresponding values of $k$ are 5 and 16, respectively, which are both viable, so the possible values of $n$ are 34 and 38.

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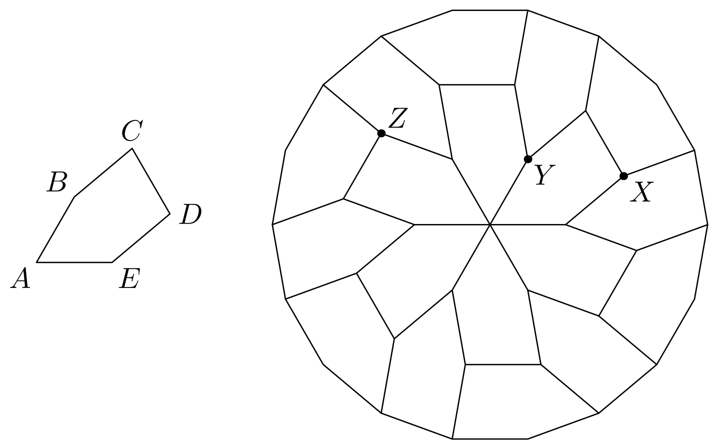
**2/1/19.** A regular 18-gon is dissected into 18 pentagons, each of which is congruent to pentagon $ABCDE$, as shown. All sides of the pentagon have the same length.




(a) Determine angles $A$, $B$, $C$, $D$, and $E$.

(b) Show that points $X$, $Y$, and $Z$ are collinear.

**Credit** This problem was proposed by Naoki Sato.

**Comments** Part (a) can be done by considering appropriate combinations of angles in the regular 18-gon. Part (b) can be done by showing that $\angle XYZ = 180°$. *Solutions edited by Naoki Sato.*

**Solution 1 by: Luyi Zhang (9/CT)**

(a) At the center of the 18-gon, six pentagons join together by their angle that corresponds to $\angle A$. Therefore, $\angle A = 360°/6 = 60°$. Since all sides of the pentagon are equal, triangle $ABE$ is equilateral and quadrilateral $BCDE$ is a rhombus.

$\angle ABC$ is an interior angle of the 18-gon, so $\angle B = \angle ABC = 160°$. Then

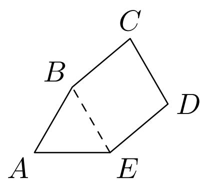
$$\angle EBC = \angle ABC - \angle ABE = 160° - 60° = 100°,$$

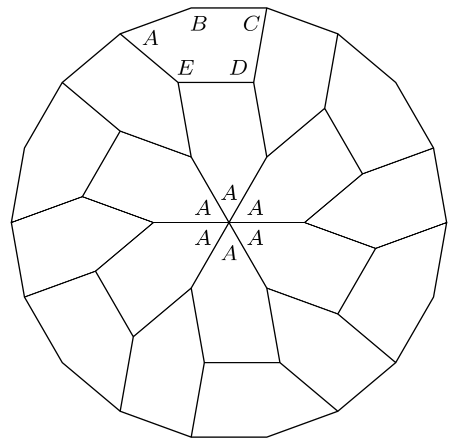
so $\angle D = \angle CDE = \angle EBC = 100°$ and

$$\angle C = \angle BED = 180° - \angle EBC = 180° - 100° = 80°.$$

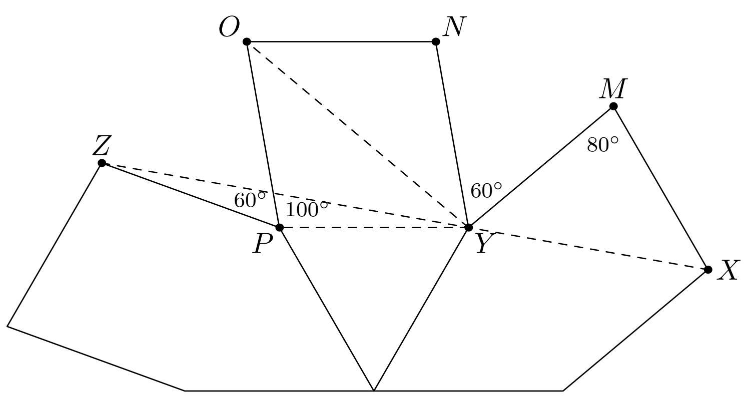
Finally, $\angle E = \angle AED = \angle AEB + \angle BED = 60° + 80° = 140°$.

To summarize, $\angle A = 60°$, $\angle B = 160°$, $\angle C = 80°$, $\angle D = 100°$, and $\angle E = 140°$.




(b) To show that points $X$, $Y$, and $Z$ are collinear we will show that $\angle XYZ = 180^\circ$. Label points $M$, $N$, $O$, and $P$, as shown below.




Since all the sides are of equal length, we can easily create isosceles triangles to assist in our angle search. In triangle $MXY$, $MX = MY$ and $\angle XMY = 80^\circ$, so $\angle MXY = \angle MYX = (180^\circ - 80^\circ)/2 = 50^\circ$.

In triangle $PYZ$, $PY = PZ$ and $\angle ZPY = \angle ZPO + \angle OPY = 60^\circ + 100^\circ = 160^\circ$, so $\angle PZY = \angle PYZ = (180^\circ - 160^\circ)/2 = 10^\circ$.

Then in triangle $OPY$, $PO = PY$ and $\angle OPY = 100^\circ$, so $\angle PYO = \angle POY = \angle NYO = \angle NOY = (180^\circ - 100^\circ)/2 = 40^\circ$, so $\angle ZYO = \angle PYO - \angle PYZ = 40^\circ - 10^\circ = 30^\circ$. Then

$$\angle XYZ = \angle MYX + \angle MYN + \angle NYO + \angle ZYO = 50^\circ + 60^\circ + 40^\circ + 30^\circ = 180^\circ,$$

and we are done.

**3/1/19.** Find all positive integers $a \le b \le c$ such that

$$\arctan\frac{1}{a} + \arctan\frac{1}{b} + \arctan\frac{1}{c} = \frac{\pi}{4}.$$

**Credit** This problem was proposed by Naoki Sato.

**Comments** First, we can use the properties of the arctan function to establish bounds on $a$. Then we can transform the given equation into an algebraic equation, from which we can deduce the solutions. *Solutions edited by Naoki Sato.*

**Solution 1 by: Damien Jiang (10/NC)**

We first establish bounds on $a$. Since $\arctan x$ is increasing on $(0, 1]$,

$$\arctan\frac{1}{a} \ge \arctan\frac{1}{b} \ge \arctan\frac{1}{c}.$$

Hence,

$$\frac{\pi}{4} = \arctan\frac{1}{a} + \arctan\frac{1}{b} + \arctan\frac{1}{c} \le 3\arctan\frac{1}{a},$$

so

$$\arctan\frac{1}{a} \ge \frac{\pi}{12} \quad \Rightarrow \quad \frac{1}{a} \ge \tan\frac{\pi}{12} = 2-\sqrt{3} \quad \Rightarrow \quad a \le \frac{1}{2-\sqrt{3}} = 2+\sqrt{3} < 4.$$

Additionally,

$$\frac{\pi}{4} = \arctan\frac{1}{a} + \arctan\frac{1}{b} + \arctan\frac{1}{c} > \arctan\frac{1}{a},$$

so

$$\frac{1}{a} < \tan\frac{\pi}{4} = 1 \quad \Rightarrow \quad a > 1.$$

Therefore, the only possible values of $a$ are $a = 2$ and $a = 3$.

From the original equation, we subtract $\arctan\frac{1}{c}$, and take the tangent of both sides to get

$$\frac{\frac{1}{a}+\frac{1}{b}}{1-\frac{1}{ab}} = \frac{1-\frac{1}{c}}{1+\frac{1}{c}}.$$

Note that this equation is equivalent with the original because $\tan x$ is injective on $(0, 1]$. Multiplying, clearing denominators, and rearranging, we get

$$abc + 1 = ab + ac + bc + a + b + c.$$

If $a = 2$, then

$$\begin{aligned} 2bc + 1 &= 2(b + c) + bc + 2 + b + c \\ \Rightarrow \quad bc - 3(b + c) &= 1 \\ \Rightarrow \quad (b - 3)(c - 3) &= 10. \end{aligned}$$

Because $c > b$, we have $b = 4$, $c = 13$ or $b = 5$, $c = 8$.

If $a = 3$, then

$$\begin{aligned} 3bc + 1 &= 3(b + c) + bc + 3 + b + c \\ \Rightarrow \quad 2bc - 4(b + c) &= 2 \\ \Rightarrow \quad (b - 2)(c - 2) &= 5. \end{aligned}$$

Because $c > b$, we have $b = 3$, $c = 7$.

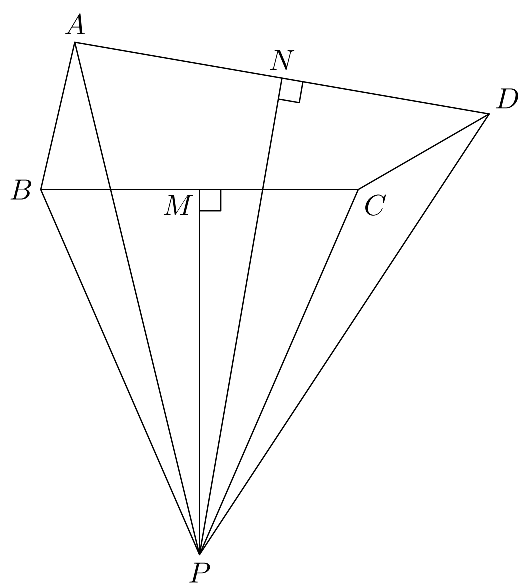
Therefore, the only solutions are $(a, b, c) = (2, 4, 13)$, $(2, 5, 8)$, and $(3, 3, 7)$.

**4/1/19.** In convex quadrilateral $ABCD$, $AB = CD$, $\angle ABC = 77°$, and $\angle BCD = 150°$. Let $P$ be the intersection of the perpendicular bisectors of $\overline{BC}$ and $\overline{AD}$. Find $\angle BPC$.

**Credit** This problem was proposed by Naoki Sato.

**Comments** Since $P$ lies on the perpendicular bisector of $BC$, $PB = PC$. This and similar observations lead to the construction of congruent triangles which determine $\angle BPC$. In addition, the solution below rigorously estalishes the location of point $P$. *Solutions edited by Naoki Sato.*

**Solution 1 by: Carl Lian (9/MA)**




Note that there are three distinct cases for the position of $P$: Either outside quadrilateral $ABCD$ on the side of $BC$, that is, $PM < PN$; outside quadrilateral $ABCD$ on the side of $AD$, that is, $PN < PM$; or inside quadrilateral $ABCD$. We first deal with the first case, and then prove that the second and third cases are impossible.

Let $M$ be the midpoint of $BC$ and $N$ the midpoint of $AD$. We have $BM = MC$ and $AN = ND$, and $\angle BMP = \angle CMP = \angle ANP = \angle DNP = 90°$, so $\triangle BMP \cong \triangle CMP$ and $\triangle ANP \cong \triangle DNP$. From these congruences, $BP = CP$ and $AP = DP$, and we are given that $AB = CD$. Therefore, $\triangle ABP \cong \triangle DCP$, and $\angle ABP = \angle DCP$.

Let $\theta = \angle CBP$. Then $\angle DCP = \angle ABP = 77° + \theta$, and $\angle BCP = \theta$. Now, by the angles around $C$, we have $\angle DCB + \angle BCP + \angle PCD = 150° + \theta + 77° + \theta = 360°$, so $2\theta = 133°$. Hence, $\angle BPC = 2\angle BPM = 2(90° - \theta) = 180° - 2\theta = 47°$.

For the second case, assume by way of contradiction that $P$ lies outside quadrilateral $ABCD$, on the side of $AD$. Again, $\triangle ABP \cong \triangle DCP$. We have $\angle PBA = \angle PCD$, and also $\angle PBM = \angle PCM$ from $\triangle PBM \cong \triangle PCM$. Adding these gives $\angle PBA + \angle PMB = \angle PCD + \angle PCM$, and thus $\angle ABC = \angle BCD$, but this is a contradiction because $\angle ABC = 77^\circ$, and $\angle BCD = 150^\circ$, so $P$ cannot lie outside quadrilateral $ABCD$ on the side of $AD$.

For the third case, assume by way of contradiction that $P$ lies inside quadrilateral $ABCD$. Again, $\triangle ABP \cong \triangle DCP$. We have $\angle PBA = \angle PCD$, and also $\angle PBM = \angle PCM$ from $\triangle PBM \cong \triangle PCM$. Adding these gives $\angle PBA + \angle PMB = \angle PCD + \angle PCM$, and thus $\angle ABC = \angle BCD$, but this is a contradiction because $\angle ABC = 77^\circ$ and $\angle DBC = 150^\circ$, so $P$ cannot lie inside quadrilateral $ABCD$.

Therefore, the first case is the only possible case, and our assertion that $\angle BPC = 47^\circ$ still holds.

**5/1/19.** Let $c$ be a real number. The sequence $a_1$, $a_2$, $a_3$, $\ldots$ is defined by $a_1 = c$ and $a_n = 2a_{n-1}^2 - 1$ for all $n \geq 2$. Find all values of $c$ such that $a_n < 0$ for all $n \geq 1$.

**Credit** This problem was proposed by Naoki Sato.

**Comments** It is not difficult to show that the value $c = -\frac{1}{2}$ works. If $c \neq -\frac{1}{2}$, then the terms of the sequence must diverge from $-\frac{1}{2}$, to the point where they become positive. The following solution uses a rigorous bounding argument. *Solutions edited by Naoki Sato.*

**Solution 1 by: Sam Elder (12/CO)**

The only value is $c = -\frac{1}{2}$.

If $a_n = -\frac{1}{2}$, then $a_{n+1} = 2a_n^2 - 1 = 2(-\frac{1}{2})^2 - 1 = -\frac{1}{2}$, so if $c = -\frac{1}{2}$, then $a_n = -\frac{1}{2} < 0$ for all $n \geq 1$ and the result is achieved.

Assume $c \neq -\frac{1}{2}$, and define the sequence $b_n = 2a_n + 1$. Assume that $a_n < 0$ for all $n$, so $b_n < 1$ for all $n$. A recursion for the $b_n$ is derived from that for the $a_n$:

$$\begin{aligned} \frac{b_n - 1}{2} &= 2\left(\frac{b_{n-1} - 1}{2}\right)^2 - 1 \\ \Rightarrow \quad b_n - 1 &= (b_{n-1} - 1)^2 - 2 \\ \Rightarrow \quad b_n &= b_{n-1}(b_{n-1} - 2) \\ \Rightarrow \quad b_n &= b_{n-2}(2 - b_{n-2})(2 - b_{n-1}) \end{aligned}$$

for all $n > 2$. If $b_n = 0$, then $b_{n-1} = 0$ or $b_{n-1} = 2$. However, by assumption, $b_n < 1$ for all $n$, and $b_1 = 2a_1 + 1 = 2c + 1 \neq 0$, so $b_n \neq 0$ for all $n$.

If $b_n < 0$, then $b_{n+1} = b_n(b_n - 2) > 0$. Likewise, if $b_n > 0$, then $b_{n+1} < 0$ since $b_n < 1$ and so $b_n - 2 < -1 < 0$. Hence, the terms $b_n$ alternate in sign, so for all $n$, one of $b_{n-1}$ and $b_{n-2}$ is negative. The other is less than 1, so

$$\frac{b_n}{b_{n-2}} = (2 - b_{n-2})(2 - b_{n-1}) > (2 - 0)(2 - 1) = 2.$$

Let $m = 1$ if $b_1$ is positive, and $m = 2$ if $b_2$ is positive, so $b_m$ is positive. Take $l$ sufficiedntly large so that $b_m > 2^{-l}$. Then

$$b_{m+2l} > 2b_{m+2(l-1)} > 2^2 b_{m+2(l-2)} > \cdots > 2^{l-1} b_{m+2} > 2^l b_m > 1,$$

a contradiction. So $c = -\frac{1}{2}$ is the only solution.