1/4/18. Let $S(n) = \sum_{i=1}^{n}(-1)^{i+1}i$. For example, $S(4) = 1 - 2 + 3 - 4 = -2$.
(a) Find, with proof, all positive integers a, b such that $S(a) + S(b) + S(a + b) = 2007$.
(b) Find, with proof, all positive integers c, d such that $S(c) + S(d) + S(c + d) = 2008$.

Credit This problem was proposed by Dave Patrick, and was based on a discussion at the 2006 World Federation of National Mathematics Competitions conference.

Comments Since both parts have the form $S(m)+S(n)+S(m+n)$, it is easiest to analyze this form first to solve for a, b, c and d. *Solutions edited by Naoki Sato.*

Solution 1 by: Sam Elder (11/CO)

If n is even, then

$$S(n) = (1 - 2) + (3 - 4) + \cdots + [(n - 1) - n] = \underbrace{-1 - 1 - \cdots - 1}_{n/2\ -1\text{s}} = -\frac{n}{2}.$$

If n is odd, then $S(n) = S(n - 1) + n = -\frac{n-1}{2} + n = \frac{n+1}{2}$. We now consider the expression $T(m, n) = S(m) + S(n) + S(m + n)$.

Case 1. Both m and n are odd. Then $m + n$ is even, so

$$T(m, n) = \frac{m + 1}{2} + \frac{n + 1}{2} - \frac{m + n}{2} = 1.$$

Case 2. Both m and n are even. Then $m + n$ is even, so

$$T(m, n) = -\frac{m}{2} - \frac{n}{2} - \frac{m + n}{2} = -m - n < 0.$$

Case 3. m is odd and n is even. Then $m + n$ is odd, so

$$T(m, n) = \frac{m + 1}{2} - \frac{n}{2} + \frac{m + n + 1}{2} = m + 1,$$

which is even.

Case 4. n is odd and m is even. Analogously with the previous case, $T(m, n) = n + 1$, which is again even.

None of these cases yield $T(m, n) = 2007$, so there are no solutions to part (a). For part (b), we can use either case 3 or 4, with the only difference being the ordering in the pairs. In Case 3, $m = 2007$ and n is even, and in Case 4, $n = 2007$ and m is even. Hence, the solutions are $(c, d) = (2007, n)$ and $(c, d) = (n, 2007)$, where n is any even positive integer.

USA Mathematical Talent Search

Solutions to Problem 2/4/18

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2/4/18. For how many integers n between 1 and 10^{2007}, inclusive, are the last 2007 digits of n and n^3 the same? (If n or n^3 has fewer than 2007 digits, treat it as if it had zeros on the left to compare the last 2007 digits.)

Credit This problem was proposed by Paul Bateman, Professor Emeritus at the University of Illinois at Urbana-Champaign.

Comments When solving a congruence with modulus m, we can look at the congruence with respect to each prime factor of m. Then, the solutions can be sewn together using the Chinese Remainder Theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (12/NJ)

Saying that the last 2007 digits of n and n^3 are the same is equivalent to saying that $n \equiv n^3 \pmod{10^{2007}}$, or

$$n(n-1)(n+1) \equiv 0 \pmod{10^{2007}}.$$

Therefore, $n(n-1)(n+1)$ is divisible by both 5^{2007} and 2^{2007}

Since only one of n, $n-1$, and $n+1$ can be divisible by 5, whichever one is divisible by 5 must also be divisible by 5^{2007}, so

$$n \equiv 0,\ 1,\ \text{or}\ -1 \pmod{5^{2007}}.$$

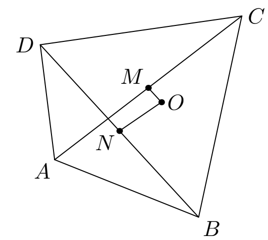
Similarly, if n is even, then both $n-1$ and $n+1$ are odd, so $n \equiv 0 \pmod{2^{2007}}$. On the other hand, if n is odd, then $(n-1)(n+1) \equiv 0 \pmod{2^{2007}}$. However, the difference between $n-1$ and $n+1$ is 2, so only one of them can be divisible by 4. Call this one $n \pm 1$. Hence, $n \mp 1$ is divisible by 2 but not 4. Therefore, $n \pm 1$ must be divisible by 2^{2006} in order that $(n-1)(n+1) \equiv 0 \pmod{2^{2007}}$, so $n \equiv \pm 1 \pmod{2^{2006}}$. Hence, if n is odd,

$$n \equiv 1,\ 2^{2006}-1,\ 2^{2006}+1,\ \text{or}\ 2^{2007}-1 \pmod{2^{2007}}.$$

Recall that if n is even, then $n \equiv 0 \pmod{2^{2007}}$.

There are three possible values of n modulo 5^{2007} and five possible values of n modulo 2^{2007}. By the Chinese Remainder Theorem, there are 15 possible values of n modulo 10^{2007}, which means there are 15 solutions n for $1 \le n \le 10^{2007}$.

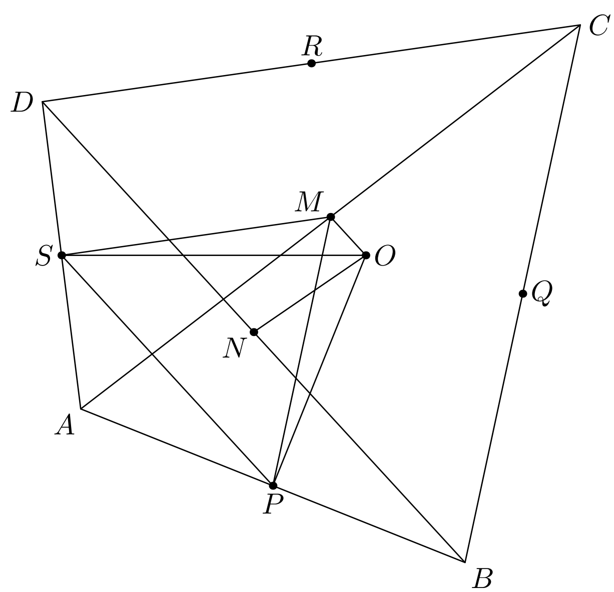
3/4/18. Let $ABCD$ be a convex quadrilateral. Let M be the midpoint of diagonal $\overline{AC}$ and N be the midpoint of diagonal $\overline{BD}$. Let O be the intersection of the line through N parallel to $\overline{AC}$ and the line through M parallel to $\overline{BD}$. Prove that the line segments joining O to the midpoints of each side of $ABCD$ divide $ABCD$ into four pieces of equal area.



Credit This problem is based on a problem from the Canadian IMO training program.

Comments This problem can be succinctly solved by using formulas for the areas of triangles involving their bases and heights. *Solutions edited by Naoki Sato.*

Solution 1 by: Kenan Diab (12/OH)



Let P, Q, R, and S be the midpoints of AB, BC, CD, and DA, respectively. Consider quadrilateral $APMS$. By definition of midpoint, we have $AB = 2AP$, $AC = 2AM$, and $AD = 2AS$. Thus, a homothecy centered at A maps quadrilateral $APMS$ to quadrilateral $ABCD$ with a factor of 2. Hence, $[APMS] = [ABCD]/4$ and $SP \parallel BD$.

But, we are given $OM \parallel BD$, so $OM \parallel SP$. Thus, O and M are the same distance from SP. Since $\triangle OSP$ and $\triangle MSP$ share side SP, it follows that $[OSP] = [MSP]$. Thus,

$$[OPAS] = [OSP] + [ASP] = [MSP] + [ASP] = [APMS] = \frac{[ABCD]}{4}.$$

Analogous homothecies centered at B, C, and D give $[OPAS] = [OQBP] = [ORCQ] = [OSDR] = [ABCD]/4$, as desired.

4/4/18. We are given a $2 \times n$ array of nodes, where n is a positive integer. A *valid connection* of the array is the addition of 1-unit-long horizontal and vertical edges between nodes, such that each node is connected to every other node via the edges, and there are no loops of any size. We give some examples for $n = 3$:

Valid Valid Invalid: loop Invalid: not connected

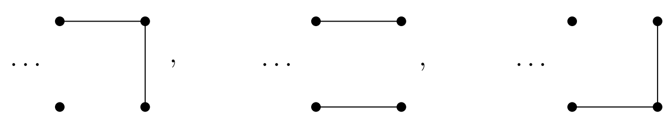
Let T_n denote the number of valid connections of the $2 \times n$ array. Find T_{10}.

Credit This problem was proposed by Naoki Sato.

Comments By constructing valid connections on a $2 \times n$ array from smaller arrays, we can obtain a recursive formula for T_n. *Solutions edited by Naoki Sato.*

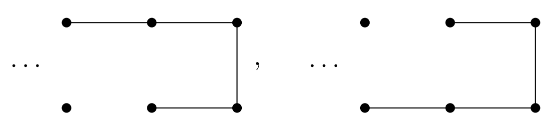
Solution 1 by: Drew Haven (11/CA)

It is trivial to note that $T_1 = 1$ because there is only one way to connect two nodes. Let us compute T_{n+1} from T_n. Given any valid connection of $2 \times n$ nodes, adding two nodes to the right gives an array of size $2 \times (n + 1)$. The additional two nodes may be connected in one of three ways:

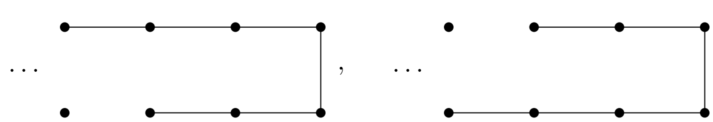


None of these result in any loops. This gives a total of $3T_n$ valid connections.

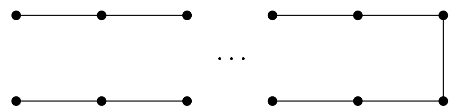
However, it is possible that a valid connection on a $2 \times (n + 1)$ array does not contain a valid connection in its leftmost $2 \times n$ nodes. Let us consider the case where the leftmost $2 \times (n - 1)$ nodes form a valid connection, but the leftmost $2 \times n$ nodes do not. There are two different ways to connect the nodes on the right to make a valid connection on a $2 \times (n + 1)$ array:



This adds $2T_{n-1}$ ways to the total count. Similarly, if only the leftmost $2 \times (n - 2)$ nodes make a valid connection, there are $2T_{n-2}$ ways:



Likewise, there are $2T_k$ ways for each $k < n$ that come from the leftmost $2 \times k$ nodes forming a valid connection. The last case to consider is the case when the leftmost two nodes are not connected, and there are no valid connections of any $2 \times k$ leftmost subarray up to $k = n$. There is only one such valid connection:



Summing these ways gives a formula for T_{n+1}:

$$T_{n+1} = 3T_n + 2T_{n-1} + 2T_{n-2} + \cdots + 2T_1 + 1. \tag{1}$$

To find a simpler recurrence, we subtract

$$T_n = 3T_{n-1} + 2T_{n-2} + \cdots + 2T_1 + 1$$

from this to give

$$\begin{aligned} T_{n+1} - T_n &= (3T_n + 2T_{n-1} + 2T_{n-2} + \cdots + 2T_1 + 1) \\ &\quad - (3T_{n-1} + 2T_{n-2} + \cdots + 2T_1 + 1) \\ &= 3T_n - T_{n-1} \\ \Rightarrow \quad T_{n+1} &= 4T_n - T_{n-1}, \end{aligned}$$

which can be rewritten as

$$T_{n+2} = 4T_{n+1} - T_n. \tag{2}$$

From (1), $T_2 = 3T_1 + 1 = 4$. Then from (2),

$$\begin{aligned} T_3 &= 4 \cdot 4 - 1 = 15, \\ T_4 &= 4 \cdot 15 - 4 = 56, \\ T_5 &= 4 \cdot 56 - 15 = 209, \\ T_6 &= 4 \cdot 209 - 56 = 780, \\ T_7 &= 4 \cdot 780 - 209 = 2911, \\ T_8 &= 4 \cdot 2911 - 780 = 10864, \\ T_9 &= 4 \cdot 10864 - 2911 = 40545, \\ T_{10} &= 4 \cdot 40545 - 10864 = 151316. \end{aligned}$$

As a side note, an explicit formula for T_n can be found using generating functions:

$$T_n = \frac{(2+\sqrt{3})^n - (2-\sqrt{3})^n}{2\sqrt{3}}. \tag{3}$$

Substituting 10 for n here gives the same answer, $T_{10} = 151316$.

5/4/18. A sequence of positive integers $(x_1, x_2, \ldots, x_{2007})$ satisfies the following two conditions:

(1) $x_n \neq x_{n+1}$ for $1 \le n \le 2006$, and

(2) $A_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$ is an integer for $1 \le n \le 2007$.

Find the minimum possible value of A_{2007}.

Credit This problem was a former proposal for the Canadian Mathematical Olympiad.

Comments Finding the optimal sequence is not difficut, but a high degree of rigor and careful reasoning must be employed to show conclusively that you have the minimum value. In particular, using a greedy algorithm is not sufficient. Both conditions (1) and (2) must be used effectively. *Solutions edited by Naoki Sato.*

Solution 1 by: Gaku Liu (11/FL)

We claim that the minimum value of A_n is $\lceil \frac{n+1}{2} \rceil$. This value is achieved for the sequence

$$x_n = \begin{cases} \dfrac{n+1}{2} & \text{for odd } n, \\[2ex] \dfrac{3n}{2} & \text{for even } n. \end{cases}$$

Indeed, if $n \ge 2$ is even, then $x_{n-1} = n/2$ and $x_{n+1} = (n+2)/2$, both of which are less than $x_n = 3n/2$. Hence, no two consecutive terms are equal, so condition (1) is satisfied. For even n,

$$\begin{aligned} A_n &= \frac{(x_1 + x_3 + \cdots + x_{n-1}) + (x_2 + x_4 + \cdots + x_n)}{n} \\ &= \frac{(1 + 2 + \cdots + n/2) + (3 + 6 + \cdots + n/2)}{n} \\ &= \frac{4(1 + 2 + \cdots + n/2)}{n} \\ &= \frac{4 \cdot n/2 \cdot (n+2)/2}{2n} \\ &= \frac{n+2}{2} = \left\lceil \frac{n+1}{2} \right\rceil, \end{aligned}$$

and for odd n,

$$\begin{aligned}
A_n &= \frac{(x_1 + x_3 + \cdots + x_{n-2}) + (x_2 + x_4 + \cdots + x_{n-1}) + x_n}{n} \\
&= \frac{[1 + 2 + \cdots + (n-1)/2] + [3 + 6 + \cdots + 3(n-1)/2] + (n+1)/2}{n} \\
&= \frac{4[1 + 2 + \cdots + (n-1)/2] + (n+1)/2}{n} \\
&= \frac{4 \cdot 1/2 \cdot (n-1)/2 \cdot (n+1)/2 + (n+1)/2}{n} \\
&= \frac{n+1}{2} = \left\lceil \frac{n+1}{2} \right\rceil .
\end{aligned}$$

We now prove this is the minimum through induction. It is true for $n = 1$, because the minimum of A_1 is $1 = \left\lceil \frac{1+1}{2} \right\rceil$. For $n = 2$, if $A_2 = 1$, then $x_1 + x_2 = 2 \Rightarrow x_1 = x_2 = 1$, which contradicts (1). Hence, the minimum of A_2 is $2 = \left\lceil \frac{2+1}{2} \right\rceil$.

Now, assume that $A_{2m} \geq \left\lceil \frac{2m+1}{2} \right\rceil = m + 1$ for some positive integer m. Let $S_n = x_1 + x_2 + \cdots + x_n$. In particular, S_n must be a multiple of n. We have $S_{2m} = 2mA_{2m} \geq 2m(m+1) = 2m^2 + 2m$. Also,

$$\begin{aligned}
& 2m^2 + m < 2m^2 + 2m < 2m^2 + 3m + 1 \\
\Rightarrow \quad & m(2m+1) < 2m^2 + 2m < (m+1)(2m+1),
\end{aligned}$$

so the least multiple of $2m + 1$ greater than $2m^2 + 2m$ is $(m+1)(2m+1)$. Since $S_{2m+1} > S_{2m} \geq 2m^2 + 2m$, we have $S_{2m+1} \geq (m+1)(2m+1)$, so

$$A_{2m+1} \geq m + 1 = \left\lceil \frac{(2m+1)+1}{2} \right\rceil .$$

Note that $2m^2 + 2m = m(2m+2)$ is a multiple of $2m+2$. The next greatest multiple of $2m+2$ is $(m+1)(2m+2)$. Suppose that $S_{2m+2} = (m+1)(2m+2) = 2m^2 + 4m + 2$. Then

$$\begin{aligned}
& 2m^2 + 3m + 1 < 2m^2 + 4m + 2 < 2m^2 + 5m + 2 \\
\Rightarrow \quad & (m+1)(2m+1) < 2m^2 + 4m + 2 < (m+2)(2m+1),
\end{aligned}$$

so the greatest multiple of $2m + 1$ less than $2m^2 + 4m + 2$ is $(m+1)(2m+1)$. Since $S_{2m+1} < S_{2m+2} = 2m^2 + 4m + 2$, we have $S_{2m+1} \leq (m+1)(2m+1)$. But we have already shown that $S_{2m+1} \geq (m+1)(2m+1)$, so $S_{2m+1} = (m+1)(2m+1)$.

Also,

$$\begin{aligned}
& 2m^2 + 2m < 2m^2 + 3m + 1 < 2m^2 + 4m \\
\Rightarrow \quad & (m+1)2m < 2m^2 + 3m + 1 < (m+2)2m,
\end{aligned}$$

so $2m^2 + 2m$ is the greatest multiple of $2m$ less than S_{2m+1}. Since $S_{2m} < S_{2m+1} = (m+1)(2m+1)$, we have $S_{2m} \le (m+1)2m$. But $S_{2m} \ge (m+1)2m$, so $S_{2m} = (m+1)2m$. Then $x_{2m+1} = S_{2m+1} - S_{2m} = (m+1)(2m+1) - (m+1)2m = m+1$, and $x_{2m+2} = S_{2m+2} - S_{2m+1} = (m+1)(2m+2) - (m+1)(2m+1) = m+1$, which contradicts (1).

Hence, $S_{2m+2} \ge (m+2)(2m+2)$, so

$$A_{2m+1} \ge m + 2 = \left\lceil \frac{(2m+2)+1}{2} \right\rceil,$$

completing the induction. Therefore, the minimum value of A_{2007} is $\left\lceil \frac{2007+1}{2} \right\rceil = 1004$.