# USA Mathematical Talent Search
Solutions to Problem 1/3/18
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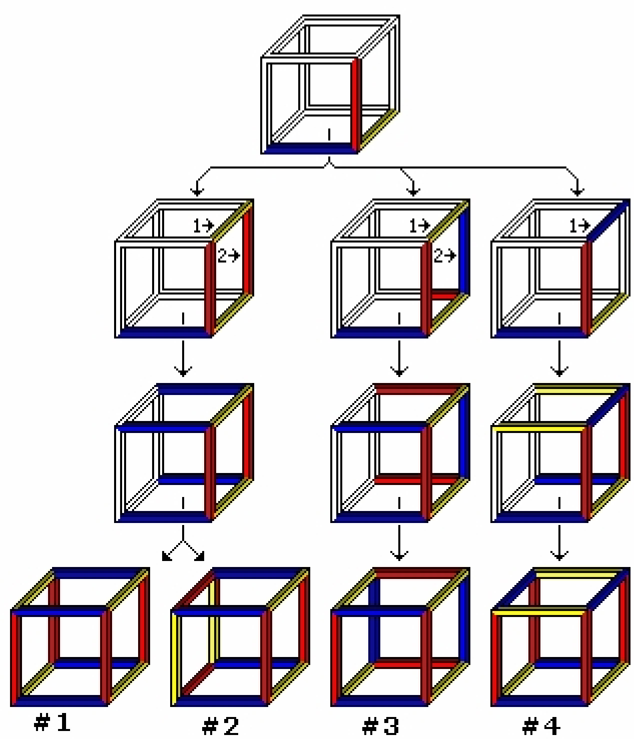
**1/3/18.** In how many distinguishable ways can the edges of a cube be colored such that each edge is yellow, red, or blue, and such that no two edges of the same color share a vertex? (Two cubes are indistinguishable if they can be rotated into positions such that the two cubes are colored exactly the same.)

**Credit** This problem was proposed by Richard Rusczyk and Sam Vandervelde.

**Comments** This problem is best solved by systematic casework and well-drawn diagrams. *Solutions edited by Naoki Sato.*

**Solution 1 by: Igor Tolkov (10/WA)**

The four possible colorings are as shown.




Without loss of generality, color three mutually adjacent edges blue, red, and yellow as shown in the top layer. Then, consider the edge marked by "1". Because an adjacent edge is red, this edge can be either yellow or blue.

First, assume that it is yellow. Then consider the edge marked by "2". This edge can be either blue or red. If it is red, then all blue edges are uniquely determined and there exist two options for the remaining four edges. This produces two colorings: One with parallel edges of the same color and one with red and yellow edges alternating. If edge 2 is blue, then the other edges are uniquely determined and we obtain a coloring with blue and red edges alternating.

Now, if edge 1 is blue, then the other edges are again uniquely determined and we obtain a coloring with blue and yellow edges alternating. This covers all cases.

Note that all four cubes have a plane of symmetry so orientation does not matter. In other words, each cube can be rotated to account for orientation.

**2/3/18.** Find, with proof, all real numbers $x$ between 0 and $2\pi$ such that

$$\tan 7x - \sin 6x = \cos 4x - \cot 7x.$$

**Credit** This problem was proposed by Marcin E. Kuczma from the University of Warsaw.

**Comments** By expressing everything in terms of sine and cosine, it is seen that the given equation is actually an inequality in disguise, where equality must occur. The equation can then be solved by finding where equality occurs. *Solutions edited by Naoki Sato.*

**Solution 1 by: Bohua Zhan (12/NJ)**

Writing everything in terms of sine and cosine and rearranging, we have:

$$\begin{aligned}
& \frac{\sin 7x}{\cos 7x} - \sin 6x = \cos 4x - \frac{\cos 7x}{\sin 7x} \\
\Leftrightarrow \quad & \frac{\sin 7x}{\cos 7x} + \frac{\cos 7x}{\sin 7x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad & \frac{\sin^2 7x + \cos^2 7x}{\sin 7x \cos 7x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad & \frac{1}{\sin 7x \cos 7x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad & \frac{2}{\sin 14x} = \cos 4x + \sin 6x \\
\Leftrightarrow \quad & 2 = \sin 14x(\cos 4x + \sin 6x).
\end{aligned}$$

Since the range of sine and cosine are $[-1, 1]$, $|\sin 14x| \leq 1$ and $|\cos 4x + \sin 6x| \leq 2$ for all $x$. Since the product of these two expressions is 2, they must all attain the maximum value. That is, $|\sin 14x| = 1$, $|\sin 6x| = 1$, and $\cos 4x = \sin 6x$. There are two cases:

**Case 1**: If $\sin 14x = -1$, then $\cos 4x = \sin 6x = -1$. So $4x = k\pi$, where $k$ is an odd integer. Then for $x$ between 0 and $2\pi$, we have $x = \frac{\pi}{4}$ , $\frac{3\pi}{4}$, $\frac{5\pi}{4}$, $\frac{7\pi}{4}$. It is not difficult to verify that only $x = \frac{\pi}{4}$ and $x = \frac{5\pi}{4}$ satisfy the other two equations.

**Case 2**: If $\sin 14x = 1$, then $\cos 4x = \sin 6x = 1$. So $4x = k\pi$, where $k$ is an even integer. For $x$ between 0 and $2\pi$, we have $x = 0$, $\frac{\pi}{2}$, $\pi$, $\frac{3\pi}{2}$, $2\pi$. Note that for all four possible values of $x$, $6x$ is a multiple of $\pi$, and $\sin 6x = 0$. Therefore, there are no solutions in this case.

In conclusion, the solutions of $x$ between 0 and $2\pi$ are $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

**3/3/18.** Three circles with radius 2 are drawn in a plane such that each circle is tangent to the other two. Let the centers of the circles be points $A$, $B$, and $C$. Point $X$ is on the circle with center $C$ such that $AX + XB = AC + CB$. Find the area of $\triangle AXB$.

**Credit** This problem was proposed by Richard Rusczyk.

**Comments** The condition $AX + XB = AC + CB$ leads to $X$ lying on an ellipse, whose equation can be determined with analytic geometry. *Solutions edited by Naoki Sato.*

**Solution 1 by: Tony Liu (12/IL)**

Position the three circles in the coordinate plane so that $A = (-2, 0)$ and $B = (2, 0)$, and let $O$ be the origin. We can easily calculate $CO = \sqrt{CA^2 - OA^2} = 2\sqrt{3}$, so let $C = (0, 2\sqrt{3})$. Now, note that the locus of all points $X$ such that $AX + XB = AC + CB$ is an ellipse $\mathcal{E}$.

More specifically, $\mathcal{E}$ is centered at $O$ and has foci at $A$ and $B$. If we let $A' = (-4, 0)$ and $B' = (4, 0)$, then $AA' + A'B = AB' + B'B = 8$, so $A'$ and $B'$ lie on the ellipse as well. The ellipse passes through $C$ as well, so the equation of $\mathcal{E}$ is given by

$$\frac{x^2}{16} + \frac{y^2}{12} = 1 \quad \Rightarrow \quad x^2 + \frac{4y^2}{3} = 16.$$

To locate $X$, we want to find the point at which the ellipse $\mathcal{E}$ intersects the circle centered at $C$, which has the equation

$$x^2 + (y - 2\sqrt{3})^2 = 4.$$

Substituting, we obtain

$$4 - (y - 2\sqrt{3})^2 + \frac{4y^2}{3} = 16 \quad \Rightarrow \quad \frac{y^2}{3} + 4\sqrt{3}y = 24.$$

Solving this quadratic and taking the positive root yields $y = 6\sqrt{5} - 6\sqrt{3}$. This is equal to the altitude of triangle $AXB$, with respect to base $AB$, so the area of this triangle is simply $\frac{1}{2} \cdot 4 \cdot y = 12\sqrt{5} - 12\sqrt{3}$.

**4/3/18.** Alice plays in a tournament in which every player plays a game against every other player exactly once. In each game, either one player wins and earns 2 points while the other gets 0 points, or the two players tie and both players earn 1 point. After the tournament, Alice tells Bob how many points she earned. Bob was not in the tournament, and does not know what happened in any individual game of the tournament.

(a) Suppose there are 12 players in the tournament, including Alice. What is the smallest number of points Alice could have earned such that Bob can deduce that Alice scored more points than at least 8 other players?

(b) Suppose there are $n$ players in the tournament, including Alice, and that Alice scored $m$ points. Find, in terms of $n$ and $k$, the smallest value of $m$ such that Bob can deduce that Alice scored more points than at least $k$ other players.

**Credit** This problem is based on a problem that appeared in *Problem Solving Journal for secondary students*, a British publication.

**Comments** To show that that the answer is $m = n + k - 1$, we must prove two things: First, we must prove that it is possible for Alice to score $n + k - 2$ points while beating at most $k - 1$ other players. Second, we must prove that if Alice scores $n + k - 1$ points, then she must have scored more points than at least $k$ other players. *Solutions edited by Naoki Sato.*

**Solution 1 by: Matt Superdock (10/PA)**

We will find a generalization in part (b) and use it to find the answer to part (a).

We claim that the smallest value of $m$ such that Bob can deduce that Alice scored more points than at least $k$ other players is $m = n + k - 1$.

If Alice scores more points than at least $k$ other players, then there are at most $n - k$ players, including Alice, that scored $m$ or more points. It is sufficient to prove that it is impossible for $n - k + 1$ players to each score $n + k - 1$ or more points, and that it is possible for $n - k + 1$ players to each score $n + k - 2$ or more points.

Suppose for the sake of contradiction that $n - k + 1$ players each score $n + k - 1$ or more points. Call these $n - k + 1$ players *good* players, and call the other $k - 1$ players *bad* players. The good players must collectively score a total of at least $(n - k + 1)(n + k - 1)$ points.

There are $\binom{n-k+1}{2}$ games between two good players, and there are two points available in each game. In these games, good players can accumulate a total of $2\binom{n-k+1}{2} = (n - k + 1)(n - k)$ points. There are $(n - k + 1)(k - 1)$ games between one good player and one bad player, and there are again two points available in each game. In these games, good players can accumulate a total of $(n-k+1)(2k-2)$ points. Good players cannot accumulate points in games between two bad players. Therefore, good players can accumulate a total of at most $(n-k+1)(n-k)+(n-k+1)(2k-2) = (n-k+1)(n+k-2)$ points, which is less than $(n-k+1)(n+k-1)$. We have reached a contradiction, so it is impossible for $n-k+1$ players to each score $n+k-1$ of more points.

It remains to be shown that it is possible for $n-k+1$ players to each score $n+k-2$ or more points. Again, call these $n-k+1$ players *good* players, and call the other $k-1$ players *bad* players. Every good player plays $n-k$ games against good players and $k-1$ games against bad players. Suppose that every game between two good players results in a tie, and every game between a good player and a bad player results in a win for the good player. In this scenario, each good player gets exactly $n-k+2(k-1) = n+k-2$ points, as desired.

Therefore the answer is indeed $m = n+k-1$.

Applying our generalization to part (a), we find that the answer is $12+8-1 = 19$.

**Further Comments**. Some students attempted to prove that if Alice scores $n+k-1$ points, then she must have scored more points than at least $k$ other players by setting up a "worst-case scenario," consisting of a number of top players that tie each other, and beat all worse players.

Proofs that simply state this worst-case scenario are not sufficiently rigorous. Proofs that began with this scenario and used the idea that for every point one player gained, another player lost, were also generally unconvincing. You must prove that Alice scores more points than at least $k$ other players in all possible scenarios, and the easiest way is as done above.

**5/3/18.** Let $f(x)$ be a strictly increasing function defined for all $x > 0$ such that $f(x) > -\frac{1}{x}$ and $f(x)f(f(x) + \frac{1}{x}) = 1$ for all $x > 0$. Find $f(1)$.

**Credit** This problem was proposed by Joe Jia.

**Comments** One important step in solving this functional equation is to substitute $f(x)+1/x$ for $x$ into the functional equation itself, a step which is suggested by the form of the functional equation. Then the strictly increasing condition can be used to solve for $f(x)$. *Solutions edited by Naoki Sato.*

**Solution 1 by: Vlad Firoiu (9/MA)**

From the given equation,

$$f\left(f(x) + \frac{1}{x}\right) = \frac{1}{f(x)}.$$

Since $y = f(x) + \frac{1}{x} > 0$ is in the domain of $f$, we have that

$$f\left(f(x) + \frac{1}{x}\right) \cdot f\left(f\left(f(x) + \frac{1}{x}\right) + \frac{1}{f(x) + \frac{1}{x}}\right) = 1.$$

Substituting $f\left(f(x) + \frac{1}{x}\right) = \frac{1}{f(x)}$ into the above equation yields

$$\frac{1}{f(x)} \cdot f\left(\frac{1}{f(x)} + \frac{1}{f(x) + \frac{1}{x}}\right) = 1,$$

so that

$$f\left(\frac{1}{f(x)} + \frac{1}{f(x) + \frac{1}{x}}\right) = f(x).$$

Since $f$ is strictly increasing, it must be 1 to 1. In other words, if $f(a) = f(b)$, then $a = b$. Applying this to the above equation gives

$$\frac{1}{f(x)} + \frac{1}{f(x) + \frac{1}{x}} = x.$$

Solving yields that

$$f(x) = \frac{1 \pm \sqrt{5}}{2x}.$$

Now, if for some $x$ in the domain of $f$,

$$f(x) = \frac{1 + \sqrt{5}}{2x},$$

then

$$f(x+1) = \frac{1 \pm \sqrt{5}}{2x+2} < \frac{1+\sqrt{5}}{2x} = f(x).$$

This contradicts the strictly increasing nature of $f$, since $x < x+1$. Therefore,

$$f(x) = \frac{1-\sqrt{5}}{2x}$$

for all $x > 0$. Plugging in $x = 1$ yields

$$f(1) = \frac{1-\sqrt{5}}{2}.$$