1/2/18. Find all positive integers n such that the sum of the squares of the digits of n is 2006 less than n.

Credit This problem was proposed by Dave Patrick.

Comments The simplest approach in this problem begins with establishing good bounds, which helps to reduce the casework. For example, with the proper bounds, you can show that n must have exactly four digits. *Solutions edited by Naoki Sato.*

Solution 1 by: Carlos Dominguez (10/OH)

We first show that n must have exactly 4 digits. Since the sum of the squares of the digits is positive, $n > 2006$. Now suppose n has d digits with $d \geq 5$. The smallest integer with d digits is 10^{d-1}, and the largest possible sum of the squares of the digits is $9^2 \cdot d = 81d$. We claim that

$$81d < 10^{d-1} - 2006.$$

When $d = 5$, we get $405 < 10000 - 2006 = 7994$, which is true. Now suppose that it holds for $d = k$, for some positive integer $k \geq 5$. Then add 81 to both sides to get

$$81(k+1) < (10^{k-1} + 81) - 2006 < 10^k - 2006,$$

which completes the induction. This implies if $d \geq 5$, then the sum of squares of the digits of n must be less than $n - 2006$. Therefore, n must have exactly 4 digits.

We want to find digits (a, b, c, d), where $n = 1000a + 100b + 10c + d$, such that $a^2 + b^2 + c^2 + d^2 = 1000a + 100b + 10c + d - 2006$. The maximum value of $a^2 + b^2 + c^2 + d^2$ is $9^2 \cdot 4 = 324$, which means that n is at most $324 + 2006 = 2330$. Since we already know that $n > 2006$, the first digit must be $a = 2$.

Substituting into our equation, we get

$$b^2 + c^2 + d^2 = 100b + 10c + d - 10.$$

Case 1: $b = 0$. We have

$$\begin{aligned} c^2 - 10c + d^2 - d &= -10 \\ \Rightarrow \quad (c-5)^2 + \left(d - \frac{1}{2}\right)^2 &= 25 + \frac{1}{4} - 10 \\ \Rightarrow \quad (2c-10)^2 + (2d-1)^2 &= 61. \end{aligned}$$

We see that 61 can be written as the sum of two squares only as $6^2 + 5^2$. This gives the solutions $(c, d) = (2, 3)$, (8,3). So two possible values of n are 2023 and 2083. Indeed, $2^2 + 0^2 + 2^2 + 3^2 + 2006 = 2023$ and $2^2 + 0^2 + 8^2 + 3^2 + 2006 = 2083$.

Case 2: $b > 0$. If $b = 1$, then

$$\begin{aligned} c^2 - 10c + d^2 - d &= 89 \\ \Rightarrow \quad (c-5)^2 + \left(d - \frac{1}{2}\right)^2 &= 89 + 25 + \frac{1}{4} \\ \Rightarrow \quad (2c-10)^2 + (2d-1)^2 &= 457. \end{aligned}$$

However, since the left hand side is at most $(2 \cdot 0 - 10)^2 + (2 \cdot 9 - 1)^2 = 389$, equality is impossible. ($(2c-10)^2$ is maximized when c is the the smallest digit 0, whereas $(2d-1)^2$ is maximized when d is the largest digit 9.) For the same reason, the cases $b = 2$ and $b = 3$ are impossible as well.

Therefore, the only answers are 2023 and 2083.

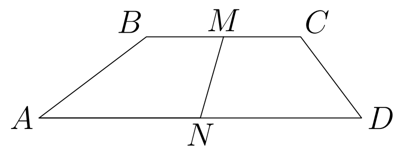
USAMTS

USA Mathematical Talent Search

Solutions to Problem 2/2/18

www.usamts.org

2/2/18. $ABCD$ is a trapezoid with $\overline{BC} \parallel \overline{AD}$, $\angle ADC = 57°$, $\angle DAB = 33°$, $BC = 6$, and $AD = 10$. M and N are the midpoints of $\overline{BC}$ and $\overline{AD}$, respectively.



(a) Find $\angle MNA$.

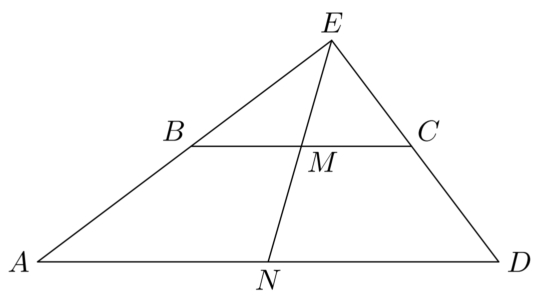
(b) Find MN.

Credit This problem was proposed by Gregory Galperin.

Comments The fact that the angles 57° and 33° are complementary is a strong indication to "complete" the right triangle, as in the solution below. Once this right triangle is constructed, the required data is easy to calculate. The fact that lines AB, CD and MN are concurrent must be justified. *Solutions edited by Naoki Sato.*

Solution 1 by: David Corwin (10/MA)

Extend AB and DC to meet at point E. Because $BC \parallel AD$, segments BC and AD are homothetic with respect to point E, with ratio $\frac{AD}{BC} = \frac{10}{6} = \frac{5}{3}$. Because M is the midpoint of BC, its corresponding homothetic point on AD is the midpoint of AD, which is N, so M and N are homothetic with respect to point E, and therefore E, M, and N are collinear.



(a) By triangle AED, $\angle AED = 180° - \angle EAD - \angle EDA = 180° - 33° - 57° = 90°$, so triangle AED is right. Because N is the midpoint of hypotenuse AD, $NE = AN$, so triangle ANE is isosceles, and $\angle NEA = \angle NAE = 33°$. Then by triangle ANE, $\angle ANE = \angle ANM = 180° - \angle NEA - \angle NAE = 180° - 33° - 33° = 114°$.

(b) Because N is the midpoint of AD, $NE = AN = \frac{AD}{2} = \frac{10}{2} = 5$. By the homothety, $ME = \frac{NE}{\frac{5}{3}} = \frac{5}{\frac{5}{3}} = 3$. Then $MN = NE - ME = 5 - 3 = 2$.

3/2/18. The expression $\lfloor x \rfloor$ means the greatest integer that is smaller than or equal to x, and $\lceil x \rceil$ means the smallest integer that is greater than or equal to x. These functions are called the *floor function* and *ceiling function*, respectively. Find, with proof, a polynomial $f(n)$ equivalent to

$$\sum_{k=1}^{n^2} \left(\left\lfloor \sqrt{k} \right\rfloor + \left\lceil \sqrt{k} \right\rceil \right)$$

for all positive integers n.

Credit This problem was proposed by Scott Kominers, a past USAMTS participant.

Comments The first thing we want to do in this sum is remove the floor and ceiling notation. Since $\sqrt{k}$ is an integer when k is a perfect square, we can consider what happens when k lies between consecutive perfect squares. Once the floor and ceiling brackets have been removed, the rest of the problem is an exercise in algebra using standard summation formula. *Solutions edited by Naoki Sato.*

Solution 1 by: Shotaro Makisumi (11/CA)

Let m be a positive integer. For $(m-1)^2+1 \le k \le m^2-1$, we have $(m-1)^2 < k < m^2 \Rightarrow m-1 < \sqrt{k} < m \Rightarrow \lfloor \sqrt{k} \rfloor + \lceil \sqrt{k} \rceil = (m-1)+m = 2m-1$. For $k = m^2$, $\lfloor \sqrt{k} \rfloor + \lceil \sqrt{k} \rceil = m+m = 2m$. Hence,

$$\begin{aligned}
\sum_{k=(m-1)^2+1}^{m^2} (\lfloor \sqrt{k} \rfloor + \lceil \sqrt{k} \rceil) &= [(m^2-1)-(m-1)^2](2m-1)+2m \\
&= (m^2-1-m^2+2m-1)(2m-1)+2m \\
&= (2m-2)(2m-1)+2m \\
&= 4m^2-4m+2,
\end{aligned}$$

which implies

$$\begin{aligned}
\sum_{k=1}^{n^2}(\lfloor\sqrt{k}\rfloor+\lceil\sqrt{k}\rceil) &= \sum_{m=1}^{n}\left[\sum_{k=(m-1)^2+1}^{m^2}(\lfloor\sqrt{k}\rfloor+\lceil\sqrt{k}\rceil)\right] \\
&= \sum_{m=1}^{n}(4m^2-4m+2) \\
&= 4\sum_{m=1}^{n}m^2 - 4\sum_{m=1}^{n}m + 2\sum_{m=1}^{n}1 \\
&= 4\cdot\frac{n(n+1)(2n+1)}{6} - 4\cdot\frac{n(n+1)}{2} + 2n \\
&= \frac{4(2n^3+3n^2+n)}{6} - \frac{12(n^2+n)}{6} + \frac{12n}{6} \\
&= \frac{8n^3+4n}{6} \\
&= \frac{4n^3+2n}{3}.
\end{aligned}$$

Therefore,

$$f(n) = \frac{4n^3+2n}{3}.$$

4/2/18. For every integer $k \geq 2$, find a formula (in terms of k) for the smallest positive integer n that has the following property:

> No matter how the elements of $\{1, 2, \ldots, n\}$ are colored red and blue, we can find k elements a_1, a_2, $\ldots$, a_k, where the a_i are not necessarily distinct elements, and an element b such that:
>
> (a) $a_1 + a_2 + \cdots + a_k = b$, and
>
> (b) all of the a_i's and b are the same color.

Credit This problem was proposed by Dave Patrick, and is a generalization of a problem that appeared on the 2004 British Mathematical Olympiad.

Comments There are two parts to this problem: You must show that for $n = k^2 + k - 2$, there is a coloring that does not satisfy the given property, and you must show that for $n = k^2 + k - 1$, any coloring satisfies the given property.

The first part can be accomplished by explicitly constructing a counter-example, and the second part can be shown by considering the colors of only a few key numbers. *Solutions edited by Naoki Sato.*

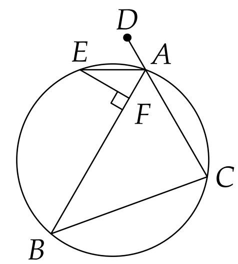
Solution 1 by: Sam Elder (11/CO)

The answer is $n = k^2 + k - 1$.

First, we show that for $n = k^2 + k - 2$, we can produce a coloring that does not satisfy these criteria. Let the numbers 1 to $k - 1$ be red, k to $k^2 - 1$ be blue, and k^2 to $k^2 + k - 2$ be red. Any k blue numbers sum to at least k^2, and all numbers at least k^2 are red. Also, if we choose k red numbers less than k, we get a total sum of at most $k(k - 1) < k^2$ but at least k, and all of these numbers are blue. Moreover, if we choose at least one red number that is at least k^2, our sum is at least $k^2 + k - 1$, which is not in our set. So no matter which k identically-colored numbers we choose, their sum is not the same color.

Now, we show that $n = k^2 + k - 1$ does work. Assume for the sake of contradiction that we cannot find $k + 1$ such integers as described in the problem. Without loss of generality, let 1 be red. Then k must be blue and k^2 must be red. Summing $k^2 + \underbrace{1 + \cdots + 1}_{k-1}$, $k^2 + k - 1$ must also be blue. Now this means $k + 1$ must be red, because otherwise we would have $k + \underbrace{(k + 1) + \cdots + (k + 1)}_{k-1} = k^2 + k - 1$, with k, $k + 1$ and $k^2 + k - 1$ blue, contradiction. But then we get $1 + \underbrace{(k + 1) + \cdots + (k + 1)}_{k-1} = k^2$, and 1, $k + 1$ and k^2 are red, contradiction. Therefore, for $n = k^2 + k - 1$, any coloring satisfies the given property.

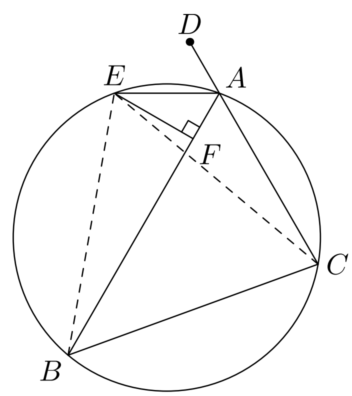
5/2/18. In triangle ABC, $AB = 8$, $BC = 7$, and $AC = 5$. We extend $\overline{AC}$ past A and mark point D on the extension, as shown. The bisector of $\angle DAB$ meets the circumcircle of $\triangle ABC$ again at E, as shown. We draw a line through E perpendicular to $\overline{AB}$. This line meets $\overline{AB}$ at point F. Find the length of $\overline{AF}$.



Credit This problem was proposed by Richard Rusczyk.

Comments An angle chase shows that triangle BEC is equilateral. Then the length of AF can be found with an application of Ptolemy's theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: Scott Kovach (11/TN)



Applying the law of cosines to triangle ABC, we see that

$$\cos \angle BAC = \frac{8^2 + 5^2 - 7^2}{2 \cdot 8 \cdot 5} = \frac{1}{2},$$

so $\angle BAC = 60^\circ$. Then $\angle EAF = \angle DAF/2 = (180^\circ - \angle BAC)/2 = 60^\circ$ as well.

Now, $\angle BEC$ subtends the same arc as $\angle BAC$, and $\angle EBC$ subtends the arc complementary to $\angle EAC$, so $\angle EBC = \angle BEC = \angle BAC = 60^\circ$, which makes triangle BEC equilateral.

Quadrilateral $EACB$ is cyclic, so by Ptolemy's theorem,

$$\begin{aligned} EA \cdot BC + EB \cdot AC &= AB \cdot EC \\ \Rightarrow \quad EA \cdot 7 + 7 \cdot 5 &= 8 \cdot 7 \\ \Rightarrow \quad EA &= 3. \end{aligned}$$

Finally, triangle EAF is a 30°-60°-90° triangle, so $AF = EA/2 = 3/2$.