1/1/18. When we perform a 'digit slide' on a number, we move its units digit to the front of the number. For example, the result of a 'digit slide' on 6471 is 1647. What is the smallest positive integer with 4 as its units digit such that the result of a 'digit slide' on the number equals 4 times the number?

Credit This problem was proposed by Naoki Sato.

Comments Let n be the number in the problem. Since the last digit of n is 4, the last digit of $4n$ is the same as the last digit of $4 \cdot 4 = 16$. But $4n$ is also the number obtained by performing a digit slide on n, so the last two digits of n are 64. One may repeat this process to find all the digits of n. *Solutions edited by Naoki Sato.*

Solution 1 by: Caroline Suen (11/CA)

Letting X be the positive integer in question, $4X$ is the result of the digit slide on X. The units digit of X is 4, and $4 \cdot 4 = 16$, so the units digit of $4X$ is 6, and the last two digits of X are 64.

We can continue the argument as follows:

$64 \cdot 4 = 256$, so the last two digits of $4X$ are 56, and the last three digits of X are 564,

$564 \cdot 4 = 2256$, so the last three digits of $4X$ are 256, and the last four digits of X are 2564,

$2564 \cdot 4 = 10256$, so the last four digits of $4X$ are 0256, and the last five digits of X are 02564,

$02564 \cdot 4 = 10256$, so the last five digits of $4X$ are 10256, and the last six digits of X are 102564, and finally

$102564 \cdot 4 = 410256$, which just happens to be the result of a digit slide on 102564.

Hence, 102564 is the smallest positive integer with 4 as its units digit such that the result of a digit slide on the number equals 4 times the number.

Solution 2 by: Howard Tong (11/GA)

Let x be the number formed by the digits other than the digit 4, and let x have k digits. Then the original number is $10x+4$, and the number obtained from the digit slide is $4 \cdot 10^k + x$. Therefore,

$$4 \cdot (10x + 4) = 4 \cdot 10^k + x,$$

which implies that

$$39x = 4 \cdot 10^k - 16.$$

The RHS is not divisible by 39 for $k = 1$, 2, 3, or 4, but when $k = 5$, $39x = 4 \cdot 10^5 - 16 = 399984 \Rightarrow x = 10256$. Therefore, the smallest possible number is 102564.

Solution 3 by: Shobhit Vishnoi (12/SC)

Let the number we are looking for be S. We have that $S = d_n d_{n-1} d_{n-2} \dots d_2 4$, where each d_k represents a digit of the decimal expansion of S. Let us construct a rational repeating decimal number N, where

$$N = 0.d_n d_{n-1} d_{n-2} \dots d_2 4 d_n d_{n-1} d_{n-2} \dots .$$

By the conditions given in the problem, $4N$ must equal $0.4 d_n d_{n-1} d_{n-2} \dots d_2 4 d_n d_{n-1} d_{n-2} \dots$.

Thus, we have the following equations:

$$N = 0.d_n d_{n-1} d_{n-2} \dots d_2 4 d_n d_{n-1} d_{n-2} \dots, \tag{1}$$
$$4N = 0.4 d_n d_{n-1} d_{n-2} \dots d_2 4 d_n d_{n-1} d_{n-2} \dots . \tag{2}$$

Multiplying equation (2) by 10, we get

$$40N = 4.d_n d_{n-1} d_{n-2} \dots d_2 4 d_n d_{n-1} d_{n-2} \dots . \tag{3}$$

Subtracting equation (1) from equation (3) gives us $39N = 4$, so

$$N = \frac{4}{39} = 0.102564102564102564\dots = 0.\overline{102564}.$$

The repeating part of N is the desired number. Therefore, $S = 102564$. Checking, we see that $4 \cdot 102564 = 410256$, and indeed satisfies the conditions.

Additional Comments. This problem resembles problem 1 from the 1962 IMO:

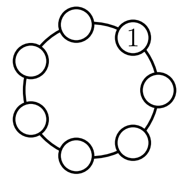
Find the smallest natural number n which has the following properties:

(a) Its decimal representation has 6 as the last digit.

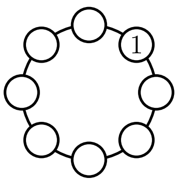
(b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n.

2/1/18.

(a) In how many different ways can the six empty circles in the diagram at right be filled in with the numbers 2 through 7 such that each number is used once, and each number is either greater than both its neighbors, or less than both its neighbors?



(b) In how many different ways can the seven empty circles in the diagram at right be filled in with the numbers 2 through 8 such that each number is used once, and each number is either greater than both its neighbors, or less than both its neighbors?



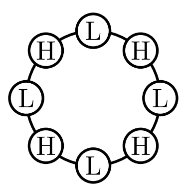
Credit This problem was proposed by Richard Rusczyk.

Comments This problem may be solved by dividing into cases, by considering which numbers must be greater than or less than their neighbors. *Solutions edited by Naoki Sato.*

Solution 1 by: Aaron Pribadi (11/AZ)

(a) Starting from a position in the loop, if the next value is greater, then the third value is less than the second, because the second value must be greater than both the first and third. If the next value is less, then the third value is greater than the second, because the second value must be less than both the first and third. This creates a higher, lower, higher, lower pattern. This pattern cannot exist with an odd number of positions; therefore, it is impossible to complete a seven-position diagram.

(b) Designate four 'high' positions as those greater than their neighbors, and four 'low' that are less than their neighbors, in an alternating pattern.



Neither 1 nor 2 may be in a 'high' position because it would require two numbers lower than it. Similarly, neither 7 nor 8 may be in a low position. So, there are $\binom{4}{2} = 6$ ways of dividing the numbers 1 to 8 into the two groups 'high' and 'low.'

Case 1. Low: 1, 2, 3, 4 High: 5, 6, 7, 8

Because all of the lows are less than all of the highs and all of the highs are greater than all of the lows, any low may be in any of the four low positions, and any high may be in any high position. Because the 1 is in four possible positions rather than one fixed position, there is a uniform overcounting by a factor of four. Therefore, for this case there are

$$\frac{4! \times 4!}{4} = 144$$

possible arrangements.

Case 2. Low: 1, 2, 3, 5 High: 4, 6, 7, 8

For a given arrangement of the lows (4! arrangements), the 4 cannot be next to the 5, so it has 2 possible positions. The other three highs may be in any of the three remaining positions and still produce a valid arrangement (3! arrangements). Therefore, there are

$$\frac{4! \times 2 \times 3!}{4} = 72$$

possible arrangements.

Case 3. Low: 1, 2, 4, 5 High: 3, 6, 7, 8

For a given arrangement of the highs, neither the low-4 nor the low-5 can be next to the high-3, so there are two positions and 2! arrangements. The other two lows may be in any of the two remaining positions and still produce a valid arrangement (2! arrangements). Therefore, there are

$$\frac{4! \times 2! \times 2!}{4} = 24$$

possible arrangements.

Case 4. Low: 1, 2, 3, 6 High: 4, 5, 7, 8

For a given arrangement of the lows, neither the high-4 nor the high-5 can be next to the low-6, so there are two positions and 2! arrangements. The other two highs may be in any of the two remaining positions and still produce a valid arrangement (2! arrangements). Therefore, there are

$$\frac{4! \times 2! \times 2!}{4} = 24$$

possible arrangements.

Case 5. Low: 1, 2, 4, 6 High: 3, 5, 7, 8

For a given position of the 6 (4 possible locations), the 3 must be in a non-adjacent location (2 possibilities). The 7 and 8 both must be next to the low-6, and the 1 and 2 both must be next to the high-3, but the 7 and 8 can switch with each other, and the 1 and 2 can also switch (2 × 2 possibilities). That leaves the low-4 and high-5, each with only one possible low or high location. Therefore, there are

$$\frac{4 \times 2 \times 2 \times 2}{4} = 8$$

possible arrangements.

Case 6. Low: 1, 2, 5, 6   High: 3, 4, 7, 8

The 7 and 8 are the only highs greater than the low 5 and 6, but the 7 and 8 cannot surround both the 5 and 6, so no arrangement is possible.

Therefore, there are a total of $144 + 72 + 24 + 24 + 8 = 272$ valid arrangements.

Solution 2 by: Gaku Liu (11/FL)

(a) Assume such a numbering exists. Let the numbers in the circles, starting from 1 going counterclockwise, be 1, a_1, a_2, $\ldots$, a_6. Since $a_1 > 1$, we must also have $a_1 > a_2$. It follows that $a_2 < a_3$, $a_3 > a_4$, and so on. In general, if i is odd, a_i is greater than its neighbors, and if i is even, a_i is less than its neighbors. Then $a_6 < 1$, a contradiction. So there are no possible numberings. This generalizes to any even number of circles.

(b) Let A_n be the answer to the general problem for n circles. We will find a recursive formula for odd n. Let the numbers in the circles, starting from 1 going counterclockwise, be 1, a_1, a_2, $\ldots$, a_{2m+1}, where m is a nonnegative integer. As in part (a), a_i is less than its neighbors if and only if i is even. Since there are no two numbers the number 2 can be greater than, $a_i = 2$ only if i is even.

Suppose $a_{2k} = 2$. Then the numbers 1 and 2 divide the larger circle into two arcs with $2k-1$ and $2m-2k+1$ circles each. The numbers 3 through 7 can be distributed between the two arcs in $\binom{2m}{2k-1}$ ways. Consider the numbers in the arc with $2k-1$ circles. Since every number is greater than 1 and 2, they can be arranged in the arc in the same number of ways as they can be arranged in the original diagram with $2k-1$ circles, which is A_{2k-1}. Similarly, the numbers in the other arc can be arranged in $A_{2m-2k+1}$ ways. Therefore, the total number of ways the empty circles can be filled in given $a_{2k} = 2$ is $\binom{2m}{2k-1}A_{2k-1}A_{2m-2k+1}$. Summing up the values for $k = 1, 2, \ldots, m$, we have

$$A_{2m+1} = \sum_{k=1}^{m} \binom{2m}{2k-1} A_{2k-1}A_{2m-2k+1}.$$

We have $A_1 = 1$, so $A_3 = \binom{2}{1}A_1^2 = 2$, $A_5 = \binom{4}{1}A_1A_3 + \binom{4}{3}A_3A_1 = 16$, and

$$A_7 = \binom{6}{1}A_1A_5 + \binom{6}{3}A_3^2 + \binom{6}{5}A_5A_1 = 6 \cdot 16 + 20 \cdot 2^2 + 6 \cdot 16 = 272.$$

Hence, the answer is 272.

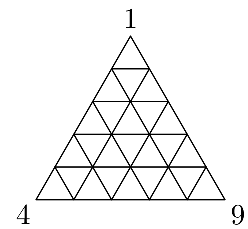
Additional Comments. Using exponential generating functions, James Sundstrom derived that the the number of arrangements for $2n$ numbers is

$$\frac{2^{2n-1}(2^{2n}-1)|B_{2n}|}{n},$$

where B_n denotes the n^{th} Bernoulli number.

3/1/18.

(a) An equilateral triangle is divided into 25 congruent smaller equilateral triangles, as shown. Each of the 21 vertices is labeled with a number such that for any three consecutive vertices on a line segment, their labels form an arithmetic sequence. The vertices of the original equilateral triangle are labeled 1, 4, and 9. Find the sum of the 21 labels.



(b) Generalize part (a) by finding the sum of the labels when there are n^2 smaller congruent equilateral triangles, and the labels of the original equilateral triangle are a, b, and c.

Credit This problem was proposed by Naoki Sato.

Comments The desired sums can be systematically computed by recognizing that for any consecutive vertices lying on a line, the labels form an arithmetic sequence. A symmetry argument can also be used to give a quick solution. *Solutions edited by Naoki Sato.*

Solution 1 by: Erik Madsen (12/CA)

This solution develops a general equation for part (b) and then applies it to solve part (a).

Without loss of generality, label the top vertex of the triangle a, the lower-left vertex b, and the lower-right vertex c. Now note that in a triangle composed of n^2 smaller triangles, we have $n+1$ horizontal rows of vertices – the first row has 1 vertex, the second has 2, and so on, with the bottom row having $n+1$ vertices. The conditions of the problem indicate that the vertices of each of these horizontal rows form an arithmetic sequence (since if any three consecutive vertices form an arithmetic sequence, then any number of consecutive vertices form an arithmetic sequence), as do the vertices of each of the diagonal edges of the triangle.

Therefore, the value of the left vertex in the m^{th} row is

$$l_m = a + \frac{b-a}{n}(m-1),$$

with m ranging from 1 to $n+1$. Similarly, the value of the right vertex in the m^{th} row is

$$r_m = a + \frac{c-a}{n}(m-1),$$

with m ranging from 1 to $n+1$. Using the formula for the sum of an arithmetic series, the sum of the values of the vertices in row m is

$$s_m = (l_m + r_m) \cdot \frac{m}{2},$$

where m ranges from 1 to $n+1$.

To find the sum S of all vertices of the triangle, we must sum s_m over all m:

$$\begin{aligned}
S &= \sum_{m=1}^{n+1} s_m = \sum_{m=1}^{n+1} (l_m + r_m) \cdot \frac{m}{2} \\
&= \sum_{m=1}^{n+1} \left[a + \frac{b-a}{n}(m-1) + a + \frac{c-a}{n}(m-1)\right] \frac{m}{2} \\
&= \sum_{m=1}^{n+1} \left[2a + \frac{b+c-2a}{n}(m-1)\right] \frac{m}{2} \\
&= \sum_{m=1}^{n+1} \left(a - \frac{b+c-2a}{2n}\right) m + \sum_{m=1}^{n+1} \left(\frac{b+c-2a}{2n}\right) m^2 \\
&= \left(a - \frac{b+c-2a}{2n}\right) \frac{(n+1)(n+2)}{2} + \left(\frac{b+c-2a}{2n}\right) \frac{(n+1)(n+2)(2n+3)}{6} \\
&= \left[a - \frac{b+c-2a}{2n} + \left(\frac{b+c-2a}{2n}\right)\left(\frac{2n+3}{3}\right)\right] \frac{(n+1)(n+2)}{2} \\
&= \left[a + \left(\frac{b+c-2a}{2n}\right)\left(\frac{2n+3}{3} - 1\right)\right] \frac{(n+1)(n+2)}{2} \\
&= \left[a + \left(\frac{b+c-2a}{2n}\right) \cdot \frac{2n}{3}\right] \frac{(n+1)(n+2)}{2} \\
&= \left(a + \frac{b+c-2a}{3}\right) \frac{(n+1)(n+2)}{3} \\
&= \frac{(a+b+c)(n+1)(n+2)}{6}.
\end{aligned}$$

Therefore,

$$S(n,a,b,c) = \frac{(a+b+c)(n+1)(n+2)}{6}.$$

Applying this formula to part (a), we have $S(5,1,4,9) = 98$.

Solution 2 by: James Sundstrom (12/NJ)

(a) Consider two triangles, both divided into 25 smaller vertices and labeled such that the labels of any three consecutive vertices on a line segment form an arithmetic sequence. Obtain a third labeled triangle by adding the corresponding labels of the original two triangles (call this operation addition of labeled triangles). The new triangle will also have the property that the labels of any three consecutive vertices on a line segment form an arithmetic sequence, because, if a_1, b_1, c_1 and a_2, b_2, c_2 are both arithmetic sequences, then

$$\begin{aligned}(c_1 + c_2) - (b_1 + b_2) &= (c_1 - b_1) + (c_2 - b_2)\\ &= (b_1 - a_1) + (b_2 - a_2)\\ &= (b_1 + b_2) - (a_1 + a_2),\end{aligned}$$

and so $a_1 + a_2$, $b_1 + b_2$, $c_1 + c_2$ is also an arithmetic sequence. It is easy to see that the sum of the labels of the new triangle is equal to the sum of the sums of the labels of the original triangles.

Say that a triangle is labeled with (a, b, c) if a is at the top vertex, b is at the bottom left vertex, and c is at bottom right vertex, and all other labels follow the arithmetic sequence rule. Add the three triangles labeled with $(1, 4, 9)$, $(4, 9, 1)$, and $(9, 1, 4)$. The result is the triangle labeled with $(14, 14, 14)$. Since, by symmetry, the original three triangles must have the same label sum, the label sum of this triangle is equal to three times the label sum of the triangle labeled with $(1, 4, 9)$. All the labels of the triangle labeled with $(14, 14, 14)$ are the same, and since the triangle in question has 21 vertices, its label sum is $21 \times 14 = 294$. Therefore the label sum of the triangle labeled with $(1, 4, 9)$ is $294/3 = 98$.

(b) Let $f_n(a, b, c)$ denote the sum of the labels of a triangle labeled with (a, b, c) when there are n^2 smaller triangles. The function f_n is additive, i.e.

$$f_n(a_1, b_1, c_1) + f_n(a_2, b_2, c_2) = f_n(a_1 + a_2, b_1 + b_2, c_1 + c_2),$$

as noted in part (a). By symmetry, $f_n(a, b, c) = f_n(b, c, a) = f_n(c, a, b)$, so

$$3f_n(a, b, c) = f_n(a, b, c) + f_n(b, c, a) + f_n(c, a, b) = f_n(a + b + c, a + b + c, a + b + c).$$

Let T_n denote the n^{th} triangular number. Then all T_{n+1} labels of the triangle labeled with $(a+b+c, a+b+c, a+b+c)$ are the same, namely $a+b+c$, so $f_n(a+b+c, a+b+c, a+b+c) = T_{n+1} \cdot (a + b + c)$. Therefore,

$$\begin{aligned}f_n(a, b, c) &= \frac{1}{3} f_n(a + b + c, a + b + c, a + b + c)\\ &= \frac{1}{3} T_{n+1}(a + b + c)\\ &= \frac{(n + 1)(n + 2)(a + b + c)}{6}.\end{aligned}$$

4/1/18. Every point in the plane is colored either red, green, or blue. Prove that there exists a rectangle in the plane such that all four of its vertices are the same color.

Credit This problem was proposed by Dave Patrick, and comes from the Carnegie Mellon Mathematical Studies Problem Seminar.

Comments Once you have the idea of using a sufficiently large grid of points, the problem quickly reduces to an application of the Pigeonhole principle. Note to USAMTS students: It is more important to get the stated problem correct, before moving onto proving a generalization. *Solutions edited by Naoki Sato.*

Solution 1 by: Adam Hesterberg (12/WA)

Consider a 4×82 rectangle of points in the plane, such as $\{(x, y) \in \mathbb{Z}^2 \mid 0 \le x \le 3,\ 0 \le y \le 81\}$. For each column, there are 4 points and 3 possible colors per point, for a total of $3^4 = 81$ possible colorings. With 82 columns, by the Pigeonhole Principle, there are two columns with the same coloring. Also, there are 4 points per column and 3 possible colors, so by the Pigeonhole Principle, some color appears twice. From each of the two columns, take some corresponding two points of a color that appears twice. These form a rectangle all of whose vertices are the same color.

Solution 2 by: Sam Elder (11/CO)

Consider a 4×19 grid of points in this plane. For each row of 4 points, by the Pigeonhole Principle, two must be the same color, for instance green. Denote such a row "green" (a row can be two colors simultaneously) and consider the colors of all 19 rows. Again by the Pigeonhole Principle, 7 must be the same color. Without loss of generality, assume this color is green.

Now consider the placement of the two green points out of four in each row. There are $\binom{4}{2} = 6$ ways to place two green points in four spots, so again by the Pigeonhole Principle, two of the seven rows must have the same placement. By choosing the four green points in those two rows, we form a monochromatic rectangle, as desired.

USA Mathematical Talent Search

Solutions to Problem 5/1/18

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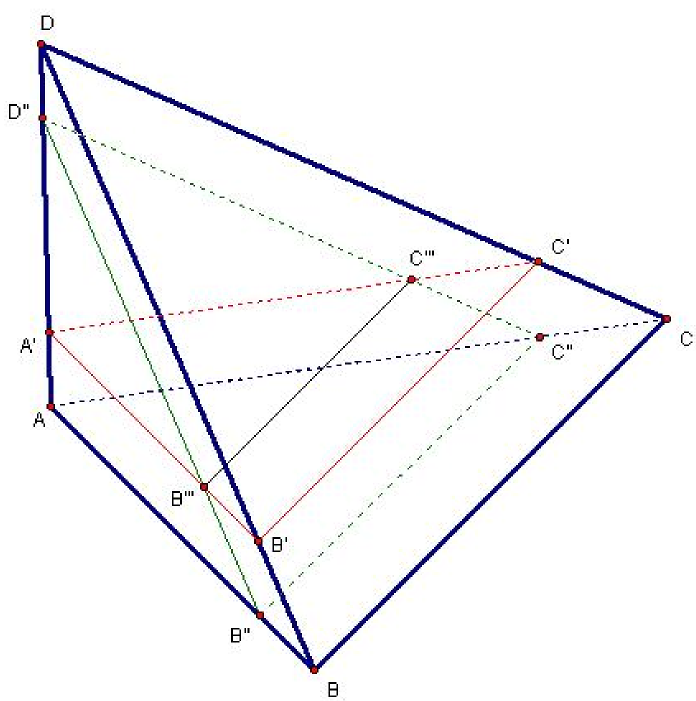
5/1/18. $ABCD$ is a tetrahedron such that $AB = 6$, $BC = 8$, $AC = AD = 10$, and $BD = CD = 12$. Plane $\mathcal{P}$ is parallel to face ABC and divides the tetrahedron into two pieces of equal volume. Plane $\mathcal{Q}$ is parallel to face DBC and also divides $ABCD$ into two pieces of equal volume. Line ℓ is the intersection of planes $\mathcal{P}$ and $\mathcal{Q}$. Find the length of the portion of ℓ that is inside $ABCD$.

Credit This problem was proposed by Richard Rusczyk.

Comments This problem is best solved by using similar tetrahedra, and drawing a nice diagram. To solve three-dimensional geometry problems, one technique that may help is to consider the two-dimenaional analogue. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (12/NJ)

Let A' denote the intersection of plane $\mathcal{P}$ and $\overline{AD}$, and define points B' and C' similarly. Let B'' denote the intersection of plane $\mathcal{Q}$ and $\overline{AB}$, and define points C'' and D'' similarly. Let B''' denote the intersection of $\mathcal{P}$, $\mathcal{Q}$, and face ABD, and let C''' denote the intersection of $\mathcal{P}$, $\mathcal{Q}$, and face ACD. Then the problem asks for the length of $\overline{B'''C'''}$.



Tetrahedrons $ABCD$ and $A'B'C'D$ are similar because plane $\mathcal{P}$ is parallel to face ABC. The volume of $ABCD$ is twice the volume of $A'B'C'D$, so $A'D = AD/\sqrt[3]{2} = 10/\sqrt[3]{2}$. Similarly, $AD'' = 10/\sqrt[3]{2}$. Since $A'D + AD'' = AD + A'D''$, we find that $A'D'' = 20/\sqrt[3]{2} - 10 = 10(\sqrt[3]{4} - 1)$.

Since plane $\mathcal{P}$ and face ABC are parallel, and plane $\mathcal{Q}$ and face DBC are parallel, tetrahedrons $ABCD$ and $A'B'''C'''D''$ are similar. Therefore,

$$B'''C''' = \frac{BC \cdot A'D''}{AD} = \frac{8 \cdot 10(\sqrt[3]{4} - 1)}{10} = 8(\sqrt[3]{4} - 1).$$