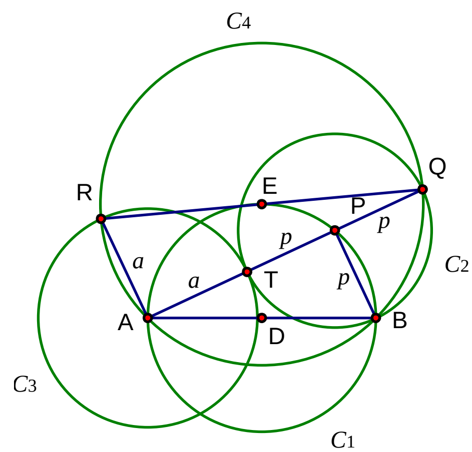
1/4/17. $\overline{AB}$ is a diameter of circle $\mathcal{C}_1$. Point P is on $\mathcal{C}_1$ such that $AP > BP$. Circle $\mathcal{C}_2$ is centered at P with radius PB. The extension of $\overline{AP}$ past P meets $\mathcal{C}_2$ at Q. Circle $\mathcal{C}_3$ is centered at A and is externally tangent to $\mathcal{C}_2$. R is on $\mathcal{C}_3$ such that $\overline{AR} \perp \overline{AQ}$. Circle $\mathcal{C}_4$ passes through A, Q, and R. Find, with proof, the ratio between the area of $\mathcal{C}_4$ and the area of $\mathcal{C}_1$, and show that this ratio is the same for all points P on $\mathcal{C}_1$ such that $AP > BP$.

Credit This problem was proposed by Ismor Fischer, University of Wisconsin.

Comments This problem can be easily solved by labeling lengths, and using right angles to apply Pythagoras's theorem. *Solutions edited by Naoki Sato.*

Solution 1 by: Matt Superdock (9/PA)

Let D be the center of C_1, and let E be the center of C_4. Let T be the point of tangency of C_2 and C_3. Let a be the radius of C_3, and let p be the radius of C_2. Then we have $AT = AR = a$ and $PB = PQ = PT = p$.



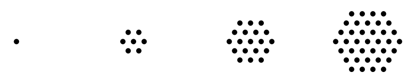
Since $\overline{AR} \perp \overline{AQ}$, $\angle RAQ$ is a right angle. Since right angles inscribe the diameter of the circle, RQ is the diameter of C_4. Since $\angle APB$ inscribes $\overline{AB}$, the diameter of C_1, $\angle APB$ is also a right angle.

The ratio we are seeking is

$$\frac{[C_4]}{[C_1]} = \frac{\pi(ER)^2}{\pi(AD)^2} = \frac{(QR)^2}{(AB)^2} = \frac{(AR)^2 + (AQ)^2}{(AP)^2 + (BP)^2} = \frac{(AR)^2 + (AT + TP + PQ)^2}{(AT + TP)^2 + (BP)^2}$$
$$= \frac{a^2 + (a + p + p)^2}{(a + p)^2 + p^2} = \frac{2a^2 + 4ap + 4p^2}{a^2 + 2ap + 2p^2} = \boxed{2}.$$

Since we made no assumptions, the ratio is 2 for all points P on C_1 such that $AP > BP$.

2/4/17. *Centered hexagonal numbers* are the numbers of dots used to create hexagonal arrays of dots. The first four centered hexagonal numbers are 1, 7, 19, and 37, as shown below.



1 7 19 37

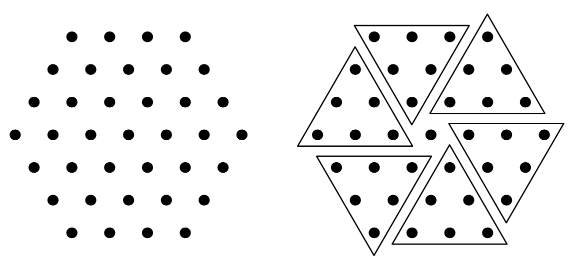
Centered Hexagonal Numbers

Consider an arithmetic sequence 1, a, b and a geometric sequence 1, c, d, where a, b, c, and d are all positive integers and $a+b=c+d$. Prove that each centered hexagonal number is a possible value of a, and prove that each possible value of a is a centered hexagonal number.

Credit This problem was proposed by Richard Rusczyk and Erin Schram.

Comments This problem requires some algebra to find the n^{th} centered hexagonal number, and then a little number theory to show the equivalence. *Solutions edited by Naoki Sato.*

Solution 1 by: Mike Nasti (11/IL)



We want to find an explicit formula for the n^{th} centered hexagonal number. Partitioning the dots as above, we see immediately that the n^{th} centered hexagonal number is 1 more than 6 times the $(n-1)^{\text{th}}$ triangular number. Thus, the n^{th} centered hexagonal number is $1+6\cdot\frac{(n-1)(n)}{2}=3n^2-3n+1$.

The arithmetic sequence 1, a, b has common difference $a-1$, so it can be written in one variable as 1, a, $2a-1$, so $b=2a-1$. The geometric sequence 1, c, d has common ratio c, so it too can be written in one variable as 1, c, c^2, so $d=c^2$. Then $a+b=c+d \Rightarrow a+2a-1=c+c^2 \Rightarrow 3a-1=c(c+1)$.

Let $c=3n-2$ for some integer n. Then c is an integer, so we know $3a-1=(3n-2)(3n-2+1)=9n^2-9n+2=3(3n^2-3n+1)-1$. Thus, whenever $c=3n-2$, $a=3n^2-3n+1$ which is the n^{th} centered hexagonal number. So each centered hexagonal number is a possible value of a.

Now, if $c(c+1) = 3a-1$ for an integer a, we can say $c(c+1) \equiv -1 \equiv 2 \pmod 3$. Then we know that $c \equiv 1 \pmod 3$ because if $c \equiv 0 \pmod 3$, then $c(c+1) \equiv 0 \pmod 3$, and if $c \equiv 2 \pmod 3$, then $c(c+1) \equiv 0 \pmod 3$. Since $c \equiv 1 \equiv -2 \pmod 3$, every possible value of c can be written in the form $3n-2$ for some integer n. Therefore the set of possible values of a is equal to the set of centered hexagonal numbers, so every possible value of a is a centered hexagonal number.

3/4/17. We play a game. The pot starts at \$0. On every turn, you flip a fair coin. If you flip heads, I add \$100 to the pot. If you flip tails, I take all of the money out of the pot, and you are assessed a "strike." You can stop the game before any flip and collect the contents of the pot, but if you get 3 strikes, the game is over and you win nothing. Find, with proof, the expected value of your winnings if you follow an optimal strategy.

Credit This problem was proposed by Dave Patrick.

Comments This problem is similar to problems 2/1/17 and 2/3/17, in that both involved expected value, and both used the technique of reducing the problem to a simpler problem. This particular problem is best solved by considering in sequence what happens when you have one strike left, then two strikes, then the full three strikes. *Solutions edited by Naoki Sato.*

Solution 1 by: Wei Hao (11/VA)

Consider first the case when I already have two strikes. In order to find the optimal time to stop, let us assume that there are x dollars in the pot. I will toss again only when I will, on average, make more than x dollars, or

$$\frac{1}{2} \cdot (x + 100) + \frac{1}{2} \cdot 0 > x,$$

which implies

$$x < 100. \tag{1}$$

So if $x < 100$, it is advantageous to risk the last strike by tossing again. But the only possible value for $x < 100$ is $x = 0$. Therefore, with two strikes, I will toss once and stop the game no matter what the outcome is. The expected return for this toss is then \$50.

Consider next when I have only one strike. I can use the same logic to decide when to stop the game, except that if I get a tail, I have one more strike to give. As a result, I should use the \$50 expected value from the above discussion as the expected winnings if I get a tail. So, assuming again that there are x dollars in the pot before a toss, it is worthwhile to risk another strike only when

$$\frac{1}{2} \cdot (x + 100) + \frac{1}{2} \cdot 50 > x,$$

which gives

$$x < 150. \tag{2}$$

To find the expected winnings in this case, I will follow the following strategy: Since immediately after a strike, there is \$0 dollar in the pot, so I will flip the coin again. If I get a head, there will be \$100 dollars in the pot, and I still have only one strike. But \$100 is less than \$150, therefore, I can flip again. On the second flip, I will either get a head and stop the game because there will be \$200 in the pot, or I will get a tail and face the two strike problem discussed before. If I get a tail on the first flip, I will also face the same two strike problem. So the expected winnings with one strike is

$$\frac{1}{2}\left(\frac{1}{2}\cdot 200+\frac{1}{2}\cdot 50\right)+\frac{1}{2}\cdot 50=87.5. \tag{3}$$

Finally, I can use the same reasoning to decide when to stop the game when I have no strikes. The only difference is that in the event of getting a tail, I must use the result of equation (3) as the expected winnings since I will have one strike then. Assuming that there are x dollars in the pot to begin with, it is advantageous to try another flip when

$$\frac{1}{2}\cdot(x+100)+\frac{1}{2}\cdot 87.5>x,$$

or

$$x<187.5. \tag{4}$$

The expected winnings for following this strategy can be calculated in the same way as the one strike case. I start by flipping the coin. If I get a head, there will be \$100 in the pot. But that is less than the \$187.5 of equation (4). So I will flip again. If I get a head again, there will be \$200 in the pot and I will stop the game. But if I get a tail on the second flip, the problem is reduced to the one strike problem with an expected winnings of \$87.5. The same thing happens if I get a tail on the first flip. Therefore, the expected winnings for this game is

$$\frac{1}{2}\left(\frac{1}{2}\cdot 200+\frac{1}{2}\cdot 87.5\right)+\frac{1}{2}\cdot 87.5=115.625. \tag{5}$$

So the expected winnings for my strategy is \$115.625.

4/4/17. Find, with proof, all irrational numbers x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational.

Credit This problem was proposed by Erin Schram.

Comments Many students were able to find the solutions $\pm\sqrt{6}$, but there are four additional solutions. One must be careful about making assumptions about irrational numbers, such as what they must look like. The best strategy here is to play the two given polynomials off of each other. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (11/IL)

We claim that $x = \pm 1 \pm \sqrt{3}$ (taking all four combinations of signs) and $x = \pm\sqrt{6}$ are the only six irrational x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational. Now, we prove that these are the only such values.

Assume we have some irrational x such that both $x^3 - 6x$ and $x^4 - 8x^2$ are rational. Let $a = x^2 - 4$, so $a^2 = x^4 - 8x^2 + 16$ is rational. Let $b = x^3 - 6x = x(x^2 - 6) = x(a - 2)$ which is also rational by hypothesis. We have

$$b^2 = x^2(a-2)^2 = (a+4)(a-2)^2 = a^3 - 12a + 16 = a(a^2 - 12) + 16.$$

In particular, because b^2 is rational, $a(a^2 - 12)$ must be rational. If $a^2 - 12 \neq 0$, then $a^2 - 12$ is rational so a must be rational as well. Otherwise, $a = \pm 2\sqrt{3}$.

If $a^2 - 12 \neq 0$ and a is rational, note that $b = x(a-2)$ is rational. Because x is irrational (and $x \neq 0$) we must have $a = 2$. Thus, $x^2 = 6$ and $x = \pm\sqrt{6}$.

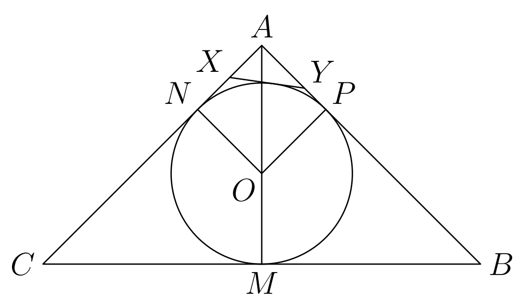
If $a = \pm 2\sqrt{3}$ then $x^2 = 4 \pm 2\sqrt{3} = (1 \pm \sqrt{3})^2$ so $x = \pm 1 \pm \sqrt{3}$ (taking all four combinations of signs). It is easily verified that all six solutions make $x^3 - 6x$ and $x^4 - 8x^2$ rational and this concludes our proof.

5/4/17. Sphere $\mathcal{S}$ is inscribed in cone $\mathcal{C}$. The height of $\mathcal{C}$ equals its radius, and both equal $12 + 12\sqrt{2}$. Let the vertex of the cone be A and the center of the sphere be B. Plane $\mathcal{P}$ is tangent to $\mathcal{S}$ and intersects segment $\overline{AB}$. X is the point on the intersection of $\mathcal{P}$ and $\mathcal{C}$ closest to A. Given that $AX = 6$, find the area of the region of $\mathcal{P}$ enclosed by the intersection of $\mathcal{C}$ and $\mathcal{P}$.

Credit This problem was proposed by Richard Rusczyk.

Comments As with most problems in three-dimensional geometry, a solution can be found by considering relevant cross-sections of the figure. Many students made incorrect assumptions about the figure, such as assuming that the altitude of the cone from A intersects the ellipse at its center. An accurately drawn figure helps prevent such errors. *Solutions edited by Naoki Sato.*

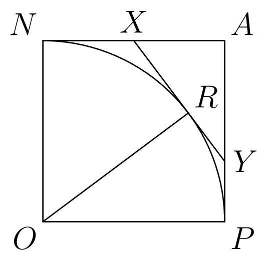
Solution 1 by: Eric Chang (11/CA)



Let Y be the furthest point on the intersection of $\mathcal{P}$ and $\mathcal{S}$ from A. Let O be the center of sphere $\mathcal{S}$. Let BC be a diameter of the base of cone $\mathcal{C}$. Let N and P be the points where the sphere $\mathcal{S}$ is tangent to AB and AC, respectively. Finally, let M be the center of the base of cone $\mathcal{C}$.

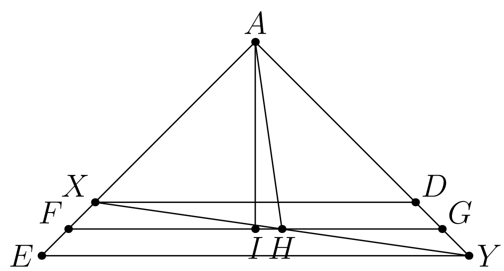
First of all, we see that the intersection of plane $\mathcal{P}$ and cone $\mathcal{C}$ will be an ellipse, which we call $\mathcal{E}$. If we take a cross section of the cone and the sphere along plane ABC, we get the above picture. Since the area of an ellipse is πab, with a and b as the semi-major and semi-minor axis, respectively, we can solve for the area if we can find the length of the major and minor axis. It is obvious that XY is the major axis of ellipse $\mathcal{E}$ since it is the longest line in the ellipse. Also, by symmetry M is the midpoint of side BC, and A, O amd M are collinear.

Because $\angle OPA$, $\angle ONA$, and $\angle NAP$ are all right angles, $\angle NOP$ must be a right angle also. Since all segments tangent to a circle from the same point have the same length, $AN = AP$ and we see that $ANOP$ is a square. Let r be the radius of $\mathcal{S}$, then we see from the picture $AM = r\sqrt{2} + r = 12 + 12\sqrt{2}$, since it is the height of the cone. Solving we get $r = 12$. Now we will consider square $ONAP$, reproduced below:

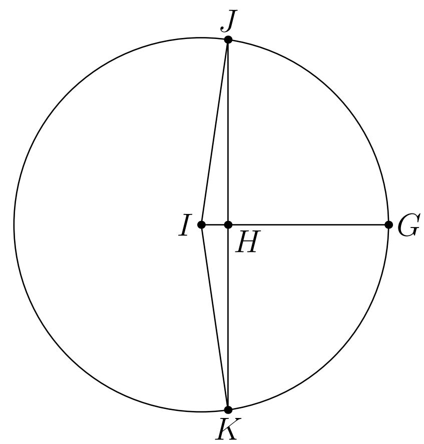


Let R be the point of tangency of XY to the circle centered at O. Since all sides of a square are congruent, they are all equal to 12, therefore, $NX = 12 - 6 = 6$. Since all segments tangent to the circle from the same point are congruent, $XR = 6$ and $RY = PY$. Labeling $AY = x$, we see that $PY = 12 - x$ and $XY = 18 - x$. Also, since AXY is a right triangle, $XY^2 = x^2 + 6^2$. Equating the two expressions for XY and solving for x, we get $x = 8$, and therefore, $AY = 8$ and $XY = 10$.

We will construct the diagram below in the following paragraph. Draw a line through X parallel to BC, and call D its point of intersection with AB, and then draw a line through Y parallel to BC also, and the point E will be its intersection with AC. Now, since these lines are parallel to the base, by similar triangles we have $AX = AD = 6$ and $AE = AY = 8$, which implies $XE = DY = 2$. If we draw line FG parallel to XD and EY and go through the midpoint of XE, it will also go through the midpoint of XY and DY because parallel lines cut all transversals in the same ratio. Call H its intersection with XY, the the minor axis will pass through this point perpendicular to XY.



Now draw the segment AI, where I is the midpoint of FG, then by a property of isosceles triangles, AI is perpendicular to FG. As a result, we can find HI by using the Pythagorean theorem on triangle AHI. Since XAY is a right triangle, by a well-known theorem, the segment AH will be congruent to XH and HY, therefore $AH = 5$. Also, using the 45-45-90 triangle AFI, we find that $AF = AX + XF = 6 + 1 = 7$. Therefore, $AI = \frac{7\sqrt{2}}{2}$ and $HI^2 = 5^2 - \left(\frac{7\sqrt{2}}{2}\right)^2 = \frac{1}{2}$, so $HI = \frac{\sqrt{2}}{2}$.



Now we take a top view of the cross section of the cone at the circle centered at I. Let JK be the minor axis. We can see that the radius of the circle is $\frac{7\sqrt{2}}{2}$ from previous calculations, and since HI is perependicular to JK, which is JH, using the Pythagorean theorem again:

$$JH^2 = \left(\frac{7\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{2}}{2}\right)^2 = 24,$$

so $JH = \sqrt{24} = 2\sqrt{6}$. Now we can plug in $2\sqrt{6}$ and $10/2 = 5$ into out formula πab. As a result, the area of ellipse $\mathcal{E}$ is $10\pi\sqrt{6}$.

Note: There is a beautiful solution using Dandelin spheres. Not only does this approach solve the problem in a nice, synthetic way, it also explains why the cross-section is an ellipse. See

`http://www.artofproblemsolving.com/Community/AoPS_Y_MJ_Transcripts.php?mj_id=128`

for a transcript of this solution.