USA Mathematical Talent Search

Solutions to Problem 1/3/17

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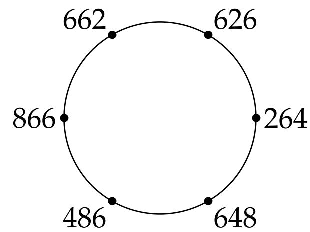
1/3/17. For a given positive integer n, we wish to construct a circle of six numbers as shown at right so that the circle has the following properties:



(a) The six numbers are different three-digit numbers, none of whose digits is a 0.

(b) Going around the circle clockwise, the first two digits of each number are the last two digits, in the same order, of the previous number.

(c) All six numbers are divisible by n.

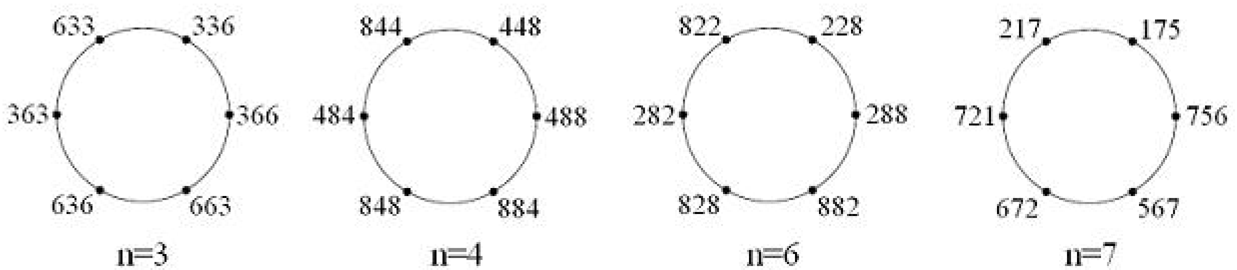
The example above shows a successful circle for $n = 2$. For each of $n = 3, 4, 5, 6, 7, 8, 9$, either construct a circle that satisfies these properties, or prove that it is impossible to do so.

Credit This problem was based on a proposal by George Berzsenyi, founder of the USAMTS.

Comments First, you must determine for each given n whether such a circle of numbers exists or not. When it exists, such a circle is not hard to find. When it does not exist, well-known divisibility rules of numbers can be used to give a rigorous proof. *Solutions edited by Naoki Sato.*

Solution 1 by: Shotaro Makisumi (10/CA)

Circles can be constructed for $n = 3$, 4, 6, and 7. An example of each is shown below.



We will show that such a construction is impossible for $n = 5$, 8, and 9.

$n = 5$: Each number must end in 0 or 5 for divisibility by 5, but 0 cannot be used, so all numbers must end in 5. Then, going around the circle, the ten digits must also all be 5, as the hundred digits. Thus, all numbers must be 555, which violates rule (a).

$n = 8$: All units digits must be even for divisibility by 8. Then the ten digits and also the hundred digits must all be even. Since 8 divides 200, 8 must also divide the last two digits. The only possibilities are $x24$, $x48$, $x64$, and $x88$, where x is an even digit. Going clockwise around the circle, $x24$ and $x64$ both force $x48$ to be the next number, which then forces $x88$ as the next number, and then 888. Thus, 888 will necessarily be repeated before the circle is complete, violating rule (a).

$n = 9$: Assume such a construction is possible, and pick a number abc (or $100a + 10b + c$) in the cycle. A number is divisible by 9 if and only if the sum of its digits equals a multiple of 9, so $9|(a + b + c)$. If we let the next number clockwise be bcd, then $9|(b + c + d)$, so $9|[(a + b + c) - (b + c + d)]$ or $9|(a - d)$. Since $1 \leq a$, $d \leq 9$, we must have $a = d$, so $bcd = bca$. Continuing clockwise, we see by the same argument that the next numbers are cab and abc. The number abc must then appear twice, which again violates rule (a).

2/3/17. Anna writes a sequence of integers starting with the number 12. Each subsequent integer she writes is chosen randomly with equal chance from among the positive divisors of the previous integer (including the possibility of the integer itself). She keeps writing integers until she writes the integer 1 for the first time, and then she stops. One such sequence is

$$12, 6, 6, 3, 3, 3, 1.$$

What is the expected value of the number of terms in Anna's sequence?

Credit This problem was proposed by Mathew Crawford.

Comments This problem is similar to problem 2/1/17, in which we also calculated an expected value using a recursive formula. *Solutions edited by Naoki Sato.*

Solution 1 by: Garrett Marcotte (12/CA)

Let (a_n) be a sequence such as described in the problem, and let $E(a_1)$ be the expected number of terms of (a_n). To calculate $E(a_1)$, suppose that d_1, d_2, …, d_k are the positive divisors of a_1, with $d_k = a_1$. Then there is a $\frac{1}{k}$ probability that any given divisor d_i is chosen as a_2. Thus, based on the method of generating the sequence, we can calculate $E(a_1)$ as follows:

$$\begin{aligned} E(a_1) &= \frac{1}{k}[E(d_1)+1] + \frac{1}{k}[E(d_2)+1] + \cdots + \frac{1}{k}[E(d_k)+1] \\ &= \frac{1}{k}[k + E(d_1) + E(d_2) + \cdots + E(d_{k-1})] + \frac{1}{k}E(d_k) \\ \Rightarrow \quad \frac{k-1}{k}E(a_1) &= \frac{1}{k}[k + E(d_1) + E(d_2) + \cdots + E(d_{k-1})] \\ \Rightarrow \quad E(a_1) &= \frac{1}{k-1}[k + E(d_1) + E(d_2) + \cdots + E(d_{k-1})]. \end{aligned}$$

Now we apply this result to find $E(12)$. By the definition of the sequence, $E(1) = 1$. The numbers 2 and 3 have the same number of divisors, namely 2, so

$$E(2) = E(3) = \frac{1}{1}[2 + E(1)] = 3.$$

The number 4 has three divisors, namely 1, 2, and 4, so

$$E(4) = \frac{1}{2}[3 + E(1) + E(2)] = \frac{7}{2}.$$

The number 6 has four divisors, namely 1, 2, 3, and 6, so

$$E(6) = \frac{1}{3}[4 + E(1) + E(2) + E(3)] = \frac{11}{3}.$$

Finall, the number 12 has six divisors, namely 1, 2, 3, 4, 6, and 12, so

$$\begin{aligned} E(12) &= \frac{1}{5}[6 + E(1) + E(2) + E(3) + E(4) + E(6)] \\ &= \frac{1}{5}\left(6 + 1 + 3 + 3 + \frac{7}{2} + \frac{11}{3}\right) \\ &= \frac{121}{30}. \end{aligned}$$

Solution 2 by: Gaku Liu (10/FL)

We will count the expected value of the number of each of the integers 12, 6, 4, 3, 2, and 1 in the sequence separately. (Note: We will use the term *decomposition* to denote a term changing from one integer to a different one.)

The integer 12 always appears as the first term of the sequence. The next integer has an equal chance of being any one of 12's six divisors, so a second 12 will appear an expected $\frac{1}{6}$ times. Then, a third 12 will appear an expected $(\frac{1}{6})^2$ times, etc., so the expected value of the number of 12's is

$$1 + \frac{1}{6} + \left(\frac{1}{6}\right)^2 + \cdots = \frac{6}{5}.$$

The integer 6 can only decompose from the integer 12. 12 has an equal chance of decomposing into any of its five proper divisors, so 6 has a $\frac{1}{5}$ chance of appearing in the sequence. 6 has four divisors, so the expected value of the number of 6's is

$$\frac{1}{5}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] = \frac{1}{5} \cdot \frac{4}{3} = \frac{4}{15}.$$

The integer 4 can only decompose from the integer 12, so it has a $\frac{1}{5}$ chance of appearing in the sequence. 4 has three divisors, so the expected value of the number of 4's is

$$\frac{1}{5}\left[1 + \frac{1}{3} + \left(\frac{1}{3}\right)^2 + \cdots\right] = \frac{1}{5} \cdot \frac{3}{2} = \frac{3}{10}.$$

The integer 3 can decompose from either 12 or 6. It has a $\frac{1}{5}$ chance of decomposing from 12. 6 has a $\frac{1}{5}$ chance of appearing in the sequence, and has three proper divisors it can decompose into, so there is a $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$ chance the integer 3 will decompose from 6. Hence, there is a $\frac{1}{5} + \frac{1}{15} = \frac{4}{15}$ chance 3 will appear in the sequence. The integer 3 has two divisors, so the expected value of the number of 3's is

$$\frac{4}{15}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{4}{15} \cdot 2 = \frac{8}{15}.$$

The integer 2 can decompose from either 12, 6, or 4. It has a $\frac{1}{5}$ chance of decomposing from 12, a $\frac{1}{5} \cdot \frac{1}{3} = \frac{1}{15}$ chance of decomposing from 6, and a $\frac{1}{5} \cdot \frac{1}{2} = \frac{1}{10}$ chance of decomposing from 4. Hence, 2 has a $\frac{1}{5} + \frac{1}{15} + \frac{1}{10} = \frac{11}{30}$ chance of appearing in the sequence. The integer 2 has two divisors, so the expected value of the number of 2's is

$$\frac{11}{30}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots\right] = \frac{11}{30} \cdot 2 = \frac{11}{15}.$$

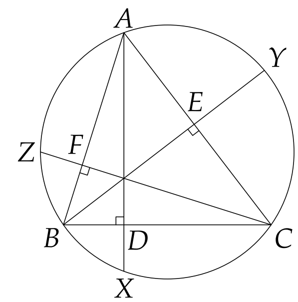
The integer 1 will always appear exactly 1 time. Hence, the expected value of the total number of terms is

$$\frac{6}{5} + \frac{4}{15} + \frac{3}{10} + \frac{8}{15} + \frac{11}{15} + 1 = \frac{121}{30}.$$

3/3/17. Points A, B, and C are on a circle such that $\triangle ABC$ is an acute triangle. X, Y, and Z are on the circle such that AX is perpendicular to BC at D, BY is perpendicular to AC at E, and CZ is perpendicular to AB at F. Find the value of

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF},$$

and prove that this value is the same for all possible A, B, C on the circle such that $\triangle ABC$ is acute.



Credit This problem was proposed by Naoki Sato.

Comments This geometry problem can be solved by recognizing that the given ratios can be expressed as ratios of certain areas, and using the fundamental result that $HD = DX$, where H is the orthocenter of triangle ABC. A solution using power of a point is also possible. *Solutions edited by Naoki Sato.*

Solution 1 by: Justin Hsu (11/CA)

Let H be the orthocenter of $\triangle ABC$. First, $\triangle BHD$ is similar to $\triangle BCE$, since they are both right triangles and they share $\angle CBE$, so $\angle BCE = \angle BHD$. Also, $\angle BXA = \angle BCA = \angle BHD$, since they both are inscribed angles that intercept the same arc BA. Now, $\triangle BXH$ is isosceles, which means that BD is the perpendicular bisector of segment HX. Therefore, $\triangle BDH \equiv \triangle BDX$, and $HD = DX$. Similarily, this can be extended to the other sides of the triangle to show that $HE = EY$ and $HF = FZ$.

Now,

$$\begin{aligned}\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} &= \frac{AD + DX}{AD} + \frac{BE + EY}{BE} + \frac{CF + FZ}{CF} \\ &= 1 + \frac{DX}{AX} + 1 + \frac{EY}{BY} + 1 + \frac{FZ}{CZ} \\ &= 3 + \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF}.\end{aligned}$$

But each fraction is a ratio between the altitudes of two triangles with the same base, so we can rewrite this sum in terms of area, where $[ABC]$ denotes the area of $\triangle ABC$:

$$\begin{aligned}3 + \frac{HD}{AD} + \frac{HE}{BE} + \frac{HF}{CF} &= 3 + \frac{[HBC]}{[ABC]} + \frac{[HCA]}{[ABC]} + \frac{[HAB]}{[ABC]} \\ &= 3 + \frac{[ABC]}{[ABC]} \\ &= 3 + 1 = 4.\end{aligned}$$

Solution 2 by: James Sundstrom (11/NJ)

By the Power of a Point Theorem,

$$AD \cdot DX = BD \cdot CD.$$

Therefore,

$$\frac{DX}{AD} = \frac{BD}{AD} \cdot \frac{CD}{AD} = \cot B \cot C.$$

Similarly,

$$\frac{EY}{BE} = \cot C \cot A,$$
$$\frac{FZ}{CF} = \cot A \cot B.$$

We can calculate

$$\begin{aligned}\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} &= \frac{AD}{AD} + \frac{DX}{AD} + \frac{BE}{BE} + \frac{EY}{BE} + \frac{CF}{CF} + \frac{FZ}{CF} \\ &= 3 + \frac{DX}{AD} + \frac{EY}{BE} + \frac{FZ}{CF} \\ &= 3 + \cot B \cot C + \cot C \cot A + \cot A \cot B \\ &= 3 + \frac{\tan A + \tan B + \tan C}{\tan A \tan B \tan C}.\end{aligned}$$

We claim that

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

for all acute triangles $\triangle ABC$ (acuteness of $\triangle ABC$ means that $\tan A$, $\tan B$, and $\tan C$ exist). [Ed: As the following argument shows, this identity holds for all triangles ABC where both sides are defined.]

We have that

$$\tan C = \tan(\pi - A - B) = \tan(-A - B) = -\tan(A + B),$$

and

$$\tan(A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$

so

$$\tan C = -\frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

This can be re-arranged to become $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

Hence,

$$\frac{AX}{AD} + \frac{BY}{BE} + \frac{CZ}{CF} = 3 + 1 = 4.$$

4/3/17. Find, with proof, all triples of real numbers (a, b, c) such that all four roots of the polynomial $f(x) = x^4 + ax^3 + bx^2 + cx + b$ are positive integers. (The four roots need not be distinct.)

Credit This problem was based on a proposal by Brian Rice.

Comments To find all possible sets of roots (which is what the problem is effectively asking for), you must use both the fact they are positive and integers. The first condition can be used to find bounds on the roots and narrow down to a finite number of cases, and the second condition can be used to find them specifically. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (11/IL)

Let p, q, r, and s be the positive integer roots of $f(x) = x^3 + ax^3 + bx^2 + cx + b$. We have

$$\begin{aligned} f(x) &= x^4 + ax^3 + bx^2 + cx + b \\ &= (x - p)(x - q)(x - r)(x - s) \\ &= x^4 - (p + q + r + s)x^3 + (pq + pr + ps + qr + qs + rs)x^2 \\ &\quad - (pqr + qrs + rsp + spq)x + pqrs. \end{aligned}$$

Comparing coefficients, we note that it suffices to find all quadruples (p, q, r, s) of positive integers such that

$$b = pqrs = pq + pr + ps + qr + qs + rs,$$

for then we obtain reals (which in fact are integers) a and c as

$$a = -(p + q + r + s) \quad \text{and} \quad c = -(pqr + qrs + rsp + spq),$$

and hence obtain all triples (a, b, c). First, let us rewrite our equation containing b by dividng through by $pqrs$. We have

$$\frac{1}{pq} + \frac{1}{pr} + \frac{1}{ps} + \frac{1}{qr} + \frac{1}{qs} + \frac{1}{rs} = 1.$$

Without loss of generality, we have $\frac{1}{rs} \geq \frac{1}{6}$ so $rs \leq 6$ for some pair of the roots (for instance, the two smallest). Assume $r \geq s$ and $p \geq q$. We now proceed with some casework.

Case 1: $rs = 6$. We either have $(r, s) = (6, 1)$ or $(3, 2)$. If $(r, s) = (6, 1)$, we have

$$6pq = pq + 7(p + q) + 6 \iff (5p - 7)(5q - 7) = 49 + 30 = 79,$$

which admittedly does not have any integer solutions (p, q) since 79 is prime and we must have $5p - 7 = 79$ and $5q - 7 = 1$, but this is clearly impossible. If $(r, s) = (3, 2)$ we have

$$6pq = pq + 5(p + q) + 6 \iff 5(pq - p - q) = 6,$$

which again does not have any integer solutions (p, q) since 6 is not divisible by 5.

Case 2: $rs = 5$. We have $(r, s) = (5, 1)$ so

$$5pq = pq + 6(p + q) + 5 \iff 2(pq - 3p - 3q) = 5,$$

which does not have any integer solutions (p, q) since 5 is odd.

Case 3: $rs = 4$. We either have $(r, s) = (4, 1)$ or $(2, 2)$. If $(r, s) = (4, 1)$, then

$$4pq = pq + 5(p + q) + 4 \iff (3p - 5)(3q - 5) = 25 + 12 = 37,$$

so $3p - 5 = 37 \Rightarrow p = 14$ and $3q - 5 = 1 \Rightarrow q = 2$, since 37 is prime. Thus, we obtain $(p, q, r, s) = (14, 2, 4, 1)$, whence $a = -21$, $b = 112$, and $c = -204$. If $(r, s) = (2, 2)$, we have

$$4pq = pq + 4(p + q) + 4 \iff (3p - 4)(3q - 4) = 16 + 12 = 28,$$

which decomposes as a product of two positive integers as $28 \cdot 1 = 14 \cdot 2 = 7 \cdot 4$. It is easily verified that only the cases $3p - 4 = 14$ and $3q - 4 = 2$ yields a valid solution $(p, q) = (6, 2)$. We obtain $(p, q, r, s) = (6, 2, 2, 2)$, whence $a = -12$, $b = 48$, and $c = -80$.

Case 4: $rs = 3$. We have $(r, s) = (3, 1)$ so

$$3pq = pq + 4(p + q) + 3 \iff 2(pq - 2p - 2q) = 3,$$

which does not have any integer solutions (p, q) since 3 is odd.

Case 5: $rs = 2$. We have $(r, s) = (2, 1)$ so

$$2pq = pq + 3(p + q) + 2 \iff (p - 3)(q - 3) = 9 + 2 = 11,$$

and we have $p - 3 = 11 \Rightarrow p = 14$ and $q - 3 = 1 \Rightarrow q = 4$. Thus, we get $(p, q, r, s) = (14, 4, 2, 1)$, which we obtained earlier in a different order with $(r, s) = (4, 1)$.

Case 6: $rs = 1$. We have $(r, s) = (1, 1)$ so

$$pq = pq + 2(p + q) + 1 \iff 2(p + q) + 1 = 0,$$

which is clearly absurd, so there are no positive integer solutions (p, q).

Thus, we have determined all desired triples (a, b, c), namely $(-21, 112, -204)$ and $(-12, 48, -80)$.

Note: The number of cases can be reduced by the following argument. First, not all of p, q, r, and s can be odd. If they were, then $pqrs$ would be odd, but then $pq+pr+ps+qr+qs+rs$, as the sum of six odd integers, would be even. Hence, at least one of them must be even.

WOLOG, let p be even. Then

$$pqrs - pq - pr - ps = p(qrs - q - r - s) = qr + qs + rs,$$

so $qr+qs+rs$ is even. If q, r, and s were all odd, then $qr+qs+rs$ would be odd, contradiction, so at least one of them must be even. WOLOG, let q be even.

Then

$$rs = pqrs - pq - pr - ps - qr - qs.$$

Each term in the RHS contains a factor of p or q, so the RHS is even. Then rs is even, so one of r and s must be even. Hence, of the four positive integers p, q, r, and s, at least three must be even. This argument, among other things, allows us to eliminate cases 2, 4, and 6 above.

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Solutions to Problem 5/3/17
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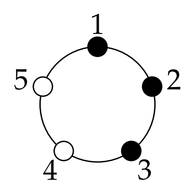
5/3/17. Lisa and Bart are playing a game. A round table has n lights evenly spaced around its circumference. Some of the lights are on and some of them off; the initial configuration is random. Lisa wins if she can get all of the lights turned on; Bart wins if he can prevent this from happening.

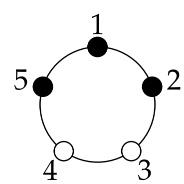
On each turn, Lisa chooses the positions at which to flip the lights, but *before* the lights are flipped, Bart, *knowing Lisa's choices*, can rotate the table to any position that he chooses (or he can leave the table as is). Then the lights in the positions that Lisa chose are flipped: those that are off are turned on and those that are on are turned off.

Here is an example turn for $n = 5$ (a white circle indicates a light that is on, and a black circle indicates a light that is off):

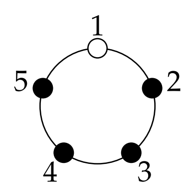
Initial Position.



Lisa says "1, 3, 4."
Bart rotates the table one position counterclockwise.



Lights in positions 1, 3, 4 are flipped.



Lisa can take as many turns as she needs to win, or she can give up if it becomes clear to her that Bart can prevent her from winning.

(a) Show that if $n = 7$ and initially at least one light is on and at least one light is off, then Bart can always prevent Lisa from winning.

(b) Show that if $n = 8$, then Lisa can always win in at most 8 turns.

Credit This problem was based on a problem from the Puzzle TOAD page at

`http://www.cs.cmu.edu/puzzle`.

They credit the problem to Ron Holzman of the Technion–Israel Institute of Technology.

Comments The case $n = 7$ can be solved by considering what moves must make Bart lose to Lisa. The case $n = 8$ can be solved by an induction argument. *Solutions edited by Naoki Sato.*

Solution 1 by: Hannah Alpert (11/CO)

For part (a), Lisa sees at least one light on and at least one light off, and she wants to turn all the lights the same color (we will say that two lights are the "same color" if they are both on or both off). For Lisa to win, the game must end with Bart being stuck: He must either turn all the off lights on or all the on lights off. Then as a last move, if all the lights are off, Lisa will turn all of them on.

However, since 7 is odd, the number of off lights cannot be the same as the number of on lights. Therefore, Bart cannot end up with such a decision; for example, if there are 3 lights on and 4 lights off, and Lisa picks three positions, Bart cannot possibly be forced to turn all the lights the same color, because it is not even possible to turn all the off lights on! Thus, for any odd n and not all lights the same color, Bart can prevent Lisa from winning.

For part (b), we will prove by induction on k that if there are $n = 2^k$ lights, Lisa can always win in at most 2^k turns. Then obviously, we can apply this fact to the case where $n = 8$.

In the base case, $k = 0$, there is one light. If it is on, Lisa wins; if it is off, Lisa requests to turn it on. That takes only one turn.

Now assume that if there are 2^k lights, then Lisa can win in at most 2^k turns. We want to show that if there are 2×2^k lights, then Lisa can win in at most 2×2^k turns. Observe that since the number of lights, 2^{k+1}, is even, each light has a partner light across from it on the table. Also notice that when Lisa chooses positions, the partners are preserved; for example, if there are 8 lights and Lisa chooses positions 1 and 5, then now matter how Bart rotates the table, she knows that some light and its partner will both change color.

Lisa's aim will be first to get each light the same color as its partner and then to turn all the pairs on. To get each light the same color as its partner, she restricts her choices such that she never chooses both a light and its partner. Since there are 2×2^k lights, there are 2^k sets of partners. Lisa imagines an equivalent table with 2^k lights, where each light corresponds to one set of partners on the big table. For each pair on the big table that has two different colors, their light is off on the small table; for each pair on the big table with the same color, their light is on at the small table. Then she makes each light the same color as its partner in 2^k moves, just as she would solve the small table in 2^k moves.

[Ed: To clarify, here is an example. Suppose the big table has 8 lights, and positions 1, 2, and 6 are on. Then the small table has 4 lights, and positions 2, 3, and 4 are on. Looking at the small table, Lisa chooses position 1. That means on the big table, she can choose position 1 or 5. After she does so and Bart rotates the table, exactly one light (among two partners) is flipped, which means that one light gets flipped on the small table. Choosing exactly one light among partners ensures that this will always be the case. Lisa continues making moves towards getting all the lights on the small table, and when that happens, for any pair of partner lights on the big table, both are on, or both are off.]

Once every light is the same color as its partner, Lisa restricts her choices such that for every light she chooses, she also chooses its partner. This time she imagines another table with 2^k lights. On this small table, each light still corresponds to one set of partners on the big table, but this time the light at the small table is the same color as the pair (since the partners now match). She finishes solving the big table in 2^k moves, just as she would solve the small table in 2^k moves. Lisa has then solved the big table with 2×2^k lights in 2×2^k moves. Our induction is complete.

Solution 2 by: Kristin Cordwell (9/NM)

(b) We identify the lights and the switching patterns with polynomials by calling the first of the consecutive eight lights (points) 1, the second x, the third x^2, and so on, all the way up to x^7. A light will be "on" if the coefficient of its respective power of x is 1; otherwise, the light will be "off". We will also think of the switching pattern being rotated, rather than the lights. Similarly, a coefficient of 1 in a switching pattern polynomial will indicate that that position is to be switched. To rotate a switching pattern n positions, we multiply the corresponding polynomial by x^n, and we take this new polynomial mod $x^8 + 1$, because $x^8 = 1$, having gone all the way around the circle. When we add and multiply polynomials, we take the coefficients mod 2, since switching (adding a power of x, with a coefficient of 1) changes an "off" (0) to an "on" (1) and vice-versa.

[Ed: For example, suppose the table has 8 lights, and positions 1, 2, and 6 are on. Then the polynomial corresponding to this set of lights is

$$x^{1-1} + x^{2-1} + x^{6-1} = 1 + x + x^5.$$

Lisa chooses positions 1, 2, and 6. Suppose Bart rotates the table by five positions. Then the lights in positions 6, 7, and 3 get flipped, so the lights in positions 1, 2, 3, and 7 are on. In terms of the polynomials, we multiply $1 + x + x^7$ by $1 + x^5$ to get

$$(1 + x + x^5)(1 + x^5) = 1 + x + 2x^5 + x^6 + x^{10}.$$

We reduce all coefficients mod 2, because flipping a light twice does nothing. Also, going around the circle, position 10 is the same as position 2, which is the same as reducing the polynomial mod $x^8 + 1$. Hence,

$$1 + x + 2x^5 + x^6 + x^{10} \equiv 1 + x + x^2 + x^6 = x^{1-1} + x^{2-1} + x^{3-1} + x^{7-1},$$

which confirms that the lights at positions 1, 2, 3, and 7 are on.]

We now define $A_0(x)$ to be the initial polynomial for the lights that are on and off, and $A_n(x)$ the polynomial for the lights at the end of the n^{th} round. We claim that if Lisa gives Bart a switching pattern equal to the "on" lights at the beginning of that round, then Lisa will win in eight or fewer rounds.

When Lisa gives Bart $A_0(x)$ for the switching pattern, we end up with

$$A_1(x) = A_0(x) + x^{a_1} A_0(x) = (1 + x^{a_1})A_0(x)$$

as the new light pattern, where Bart has rotated the switching pattern by a_1 positions. Similarly,

$$A_2(x) = (1 + x^{a_2})A_1(x),$$

where Bart rotates the second switching pattern by a_2 positions. Continuing, we get

$$\begin{aligned} A_3(x) &= (1 + x^{a_3})A_2(x), \\ A_4(x) &= (1 + x^{a_4})A_3(x), \\ &\ldots, \\ A_8(x) &= (1 + x^{a_8})A_7(x). \end{aligned}$$

Note that if $x = 1$, then for any i,

$$1 + x^{a_i} = 1 + 1^{a_i} = 2 \equiv 0,$$

so $1 + x$ is a factor of $1 + x^{a_i}$. Indeed,

$$\begin{aligned} &(1 + x)(1 + x + x^2 + \cdots + x^{a_i - 1}) \\ &= 1 + x + x^2 + \cdots + x^{a_i-1} + x + x^2 + \cdots + x^{a_i} \\ &= 1 + 2x + 2x^2 + \cdots + 2x^{a_i-1} + x^{a_i} \\ &\equiv 1 + x^{a_i}. \end{aligned}$$

Thus, setting $P_i(x) = 1 + x + x^2 + \cdots + x^{a_i-1}$, we can write

$$1 + x^{a_i} \equiv (1 + x)P_i(x).$$

So,

$$\begin{aligned} A_8(x) &= (1 + x^{a_8})A_7(x) \\ &= (1 + x^{a_8})(1 + x^{a_7})A_6(x) \\ &= \cdots \\ &= (1 + x^{a_8})(1 + x^{a_7}) \cdots (1 + x^{a_1})A_0(x) \\ &\equiv (1 + x)P_8(x)(1 + x)P_7(x) \cdots (1 + x)P_1(x)A_0(x) \\ &= (1 + x)^8 P_8(x)P_7(x) \cdots P_1(x)A_0(x). \end{aligned}$$

Now,

$$\begin{aligned}(1+x)^8 &= 1 + 8x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8 \\ &\equiv 1 + x^8 \quad \text{(reducing the coefficients mod 2)} \\ &\equiv 0, \quad \text{(reducing the polynomial mod } 1 + x^8\text{)}\end{aligned}$$

so $A_8(x) \equiv 0$. Therefore, all lights will always be off after eight rounds. Looking back at the equation

$$A_8(x) = (1 + x^{a_8})A_7(x),$$

we see that the polynomial $A_7(x)$ must then have the property that for any a_8, multiplying $A_7(x)$ by $1 + x^{a_8}$ gives the zero polynomial. The only polynomials $A_7(x)$ that have this property are $A_7(x) = 0$ and $A_7(x) = 1 + x + x^2 + \cdots + x^7$. Thus, after seven turns, the lights are either all on or all off. If the lights are all on, Lisa has won at the end of the seventh round (or sooner, if they were all on sooner). If they are all off, Lisa gives Bart the switching pattern of change everything, and Lisa wins at the end of the eighth round (or sooner, if they were all off sooner).

This generalizes to all n that are powers of two, so that Lisa can always win in at most n rounds.