USA Mathematical Talent Search
Solutions to Problem 1/2/17
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1/2/17. At the right is shown a 4×4 grid. We wish to fill in the grid such that each row, each column, and each 2×2 square outlined by the thick lines contains the digits 1 through 4. The first row has already been filled in. Find, with proof, the number of ways we can complete the rest of the grid.

1	2	3	4

Credit This problem was proposed by George Berzsenyi, founder of the USAMTS.

Comments This is a relatively simple counting problem (inspired by the latest Sudoku puzzles and simplified to a 4×4 grid), where you need a little care to make sure that you have covered all the cases and that you haven't counted any grids twice. Lynnelle Ye shows a particularly nice approach. *Solutions edited by Naoki Sato.*

Solution 1 by: Lynnelle Ye (8/CA)

The numbers in the second row must be one of the following: 3 4 1 2, 4 3 2 1, 3 4 2 1, or 4 3 1 2.

In the first two cases, the first and third columns are missing the same numbers, and the second and fourth columns are missing the same numbers, so the number chosen for the third row, first column determines the number chosen for the third row, third column, and the number chosen for the third row, second column determines the number chosen for the third row, fourth column. Once the third row is chosen, the fourth row is determined. For example, with the first case for the second row,

1	2	3	4
3	4	1	2
2 or 4	1 or 3	4 or 2	3 or 1

So these two cases contribute a total of $2^3 = 8$ possibilities.

In the second two cases, the number chosen for the third row, first column determines the number chosen for the third row, either third or fourth column. This then determines the number in the third row, second column, which then determines the third row, whatever column is left. Again, the fourth row is determined. For example, with the third case for the second row and the third row, first column equal to 2,

1	2	3	4
3	4	2	1
2	1	4	3

The third row, fourth column must be 3, which means that the third row, second column must be 1, which means that the third row, third column must be 4. So these two cases contribute a total of $2^2 = 4$ possibilities.

Therefore, there are a total of $8 + 4 = 12$ ways to complete the rest of the grid.

2/2/17. Write the number

$$\frac{1}{\sqrt{2}-\sqrt[3]{2}}$$

as the sum of terms of the form 2^q, where q is rational. (For example, $2^1 + 2^{-1/3} + 2^{8/5}$ is a sum of this form.) Prove that your sum equals $1/(\sqrt{2}-\sqrt[3]{2})$.

Credit This problem was proposed by Sidney Kravitz.

Comments This problem is effectively an exercise in rationalizing the denominator, but the twist is the presence of both the square root and the cube root. A knowledge of basic algebraic identities can take care of both. James Sundstrom arrives at the answer using two steps, Justin Hsu shows how to the same calculation in one step, and Jeffrey Manning provides a clever solution using geometric series. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (11/NJ)

The identities $(a-b)(a+b) = a^2 - b^2$ and $(a-b)(a^2+ab+b^2) = a^3 - b^3$ suggest the following approach for rationalizing the denominator. For example, setting $a = \sqrt{2}$ and $b = \sqrt[3]{2}$ gives

$$(\sqrt{2}-\sqrt[3]{2})(\sqrt{2}+\sqrt[3]{2}) = 2 - \sqrt[3]{4},$$

so

$$\frac{1}{\sqrt{2}-\sqrt[3]{2}} = \frac{\sqrt{2}+\sqrt[3]{2}}{(\sqrt{2}-\sqrt[3]{2})(\sqrt{2}+\sqrt[3]{2})} = \frac{\sqrt{2}+\sqrt[3]{2}}{2-\sqrt[3]{4}}.$$

Then setting $a = 2$ and $b = \sqrt[3]{4}$ gives

$$(2-\sqrt[3]{4})(4+2\sqrt[3]{4}+2\sqrt[3]{2}) = 8-4 = 4,$$

so

$$\begin{aligned}
\frac{\sqrt{2}+\sqrt[3]{2}}{2-\sqrt[3]{4}} &= \frac{(\sqrt{2}+\sqrt[3]{2})(4+2\sqrt[3]{4}+2\sqrt[3]{2})}{(2-\sqrt[3]{4})(4+2\sqrt[3]{4}+2\sqrt[3]{2})} \\
&= \frac{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}{4} \\
&= \sqrt{2}+\sqrt[3]{2}+\frac{\sqrt[3]{4}\sqrt{2}}{2}+1+\frac{\sqrt[3]{2}\sqrt{2}}{2}+\frac{\sqrt[3]{4}}{2} \\
&= 2^{\frac{1}{2}}+2^{\frac{1}{3}}+2^{\frac{2}{3}}\left(2^{\frac{1}{2}}\right)\left(2^{-1}\right)+2^0+2^{\frac{1}{3}}\left(2^{\frac{1}{2}}\right)\left(2^{-1}\right)+2^{\frac{2}{3}}\left(2^{-1}\right) \\
&= 2^{\frac{1}{2}}+2^{\frac{1}{3}}+2^{\frac{1}{6}}+2^0+2^{-\frac{1}{6}}+2^{-\frac{1}{3}}.
\end{aligned}$$

Solution 2 by: Justin Hsu (11/CA)

We have the identity

$$(a-b)(a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5)=a^6-b^6.$$

Now, we let $a=\sqrt{2}$ and $b=\sqrt[3]{2}$, and multiply the numerator and denominator by $a^5+a^4b+a^3b^2+a^2b^3+ab^4+b^5$. This gives us

$$\begin{aligned}&\frac{1}{\sqrt{2}-\sqrt[3]{2}}\cdot\frac{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}\\&=\frac{4\sqrt{2}+4\sqrt[3]{2}+2\sqrt[3]{4}\sqrt{2}+4+2\sqrt[3]{2}\sqrt{2}+2\sqrt[3]{4}}{4}\\&=2^{\frac{1}{2}}+2^{\frac{1}{3}}+2^{\frac{1}{6}}+2^0+2^{-\frac{1}{6}}+2^{-\frac{1}{3}},\end{aligned}$$

giving us the required sum of rational powers of 2.

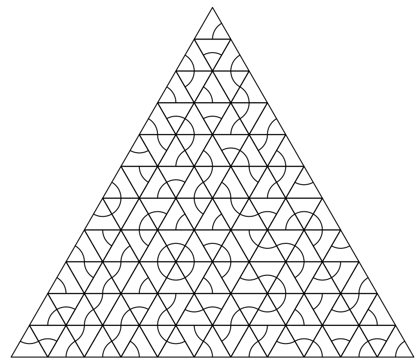
Solution 3 by: Jeffrey Manning (10/CA)

Notice that if $x\neq 0$, then the sum $x^3+x^2+x+1+x^{-1}+x^{-2}$ is a geometric series with common ratio x^{-1} and initial term x^3. If $x\neq 1$, then we can use the formula for a geometric series to get

$$x^3+x^2+x+1+x^{-1}+x^{-2}=\frac{x^3(1-x^{-6})}{1-x^{-1}}=\frac{x^3-x^{-3}}{1-x^{-1}}=\frac{(x^3-x^{-3})x^3}{(1-x^{-1})x^3}=\frac{x^6-1}{x^3-x^2}.$$

Applying this formula with $x=2^{1/6}$ gives

$$\begin{aligned}\frac{1}{\sqrt{2}-\sqrt[3]{2}}&=\frac{2^1-1}{2^{1/2}-2^{1/3}}\\&=\frac{(2^{1/6})^6-1}{(2^{1/6})^3-(2^{1/6})^2}\\&=(2^{1/6})^3+(2^{1/6})^2+2^{1/6}+1+(2^{1/6})^{-1}+(2^{1/6})^{-2}\\&=2^{1/2}+2^{1/3}+2^{1/6}+2^0+2^{-1/6}+2^{-1/3}.\end{aligned}$$



3/2/17. An equilateral triangle is tiled with n^2 smaller congruent equilateral triangles such that there are n smaller triangles along each of the sides of the original triangle. The case $n = 11$ is shown at right. For each of the small equilateral triangles, we randomly choose a vertex V of the triangle and draw an arc with that vertex as center connecting the midpoints of the two sides of the small triangle with V as an endpoint. Find, with proof, the expected value of the number of full circles formed, in terms of n.

Credit This problem was proposed by Richard Rusczyk.

Comments Trying to count in how many cases the circles appear gets very complicated, as these cases are not independent. (In other words, whether a circle appears at one vertex affects whether a circle can appear at adjacent vertices.) However, a fundamental property of expected value is that the expected value of a sum is simply the sum of the expected values, a property mentioned in our Expected Value article. Once you see this, the problem actually becomes quite easy. *Solutions edited by Naoki Sato.*

Solution 1 by: Derrick Sund (12/WA)

Consider a vertex in such a triangle that has six small triangles around it. Each of these triangles has a $\frac{1}{3}$ probability of its arc being the arc of a circle centered on that vertex. Therefore, the probability that that vertex has a full circle around it is $\frac{1}{3^6} = \frac{1}{729}$. To get the expected value of the number of full circles, we simply need to multiply this probability by the number of vertices that can have full circles around them. If the original triangle is divided into n^2 smaller triangles, then the number of such vertices will be

$$(n-2) + (n-3) + (n-4) + \cdots + 2 + 1 = \frac{(n-1)(n-2)}{2},$$

so the desired expected value is

$$\frac{(n-1)(n-2)}{2} \cdot \frac{1}{729} = \frac{(n-1)(n-2)}{1458}.$$

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Solutions to Problem 4/2/17
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4/2/17. A teacher plays the game "Duck-Goose-Goose" with his class. The game is played as follows: All the students stand in a circle and the teacher walks around the circle. As he passes each student, he taps the student on the head and declares her a 'duck' or a 'goose'. Any student named a 'goose' leaves the circle immediately. Starting with the first student, the teacher tags students in the pattern: duck, goose, goose, duck, goose, goose, etc., and continues around the circle (re-tagging some former ducks as geese) until only one student remains. This remaining student is the winner.

For instance, if there are 8 students, the game proceeds as follows: student 1 (duck), student 2 (goose), student 3 (goose), student 4 (duck), student 5 (goose), student 6 (goose), student 7 (duck), student 8 (goose), student 1 (goose), student 4 (duck), student 7 (goose) and student 4 is the winner. Find, with proof, all values of n with $n > 2$ such that if the circle starts with n students, then the n^{th} student is the winner.

Credit This problem was proposed by Mathew Crawford.

Comments Adam Hesterberg nicely solves the problem by modifying the game slightly to one that has a recursive nature. *Solutions edited by Naoki Sato.*

Solution 1 by: Adam Hesterberg (11/WA)

We claim that the n^{th} student is the winner if and only if n of the form $3^k - 2$ or $2 \cdot 3^k - 2$, where k is a positive integer.

Suppose that the game was actually "Goose-Goose-Duck," and there were two more people at the start of the circle. This game is equivalent to "Duck-Goose-Goose," since the two new people will be tagged immediately, and the pattern is equivalent to "Duck-Goose-Goose" from there. In the first round, people numbered with multiples of 3 survive, so either the winner is a multiple of 3, or there are only two people left and he is one of them.

In the latter case, the game ends with that round. In the former case, if the winner is a multiple of 3, then the first two people in the next round will be tagged, and the third will live. In general, if the last person survived the $(k-1)^{\text{st}}$ round, then the survivors of the k^{th} round will be the multiples of 3^k. Therefore, for the last person to be the final survivor, he must have the greatest power of 3 as a factor. In case there are two numbers with the same greatest power of 3 as a factor, the lower number gets tagged earlier, so the last person would still win. Therefore, the possible numbers for the last person are 3^k and $2 \cdot 3^k$–greater multiples of 3^k would produce someone with a greater power of 3. Subtract 2 to revert to the original game, getting $n = 3^k - 2$ or $n = 2 \cdot 3^k - 2$.

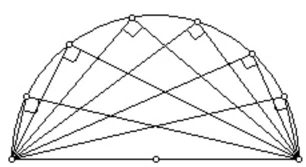
5/2/17. Given acute triangle $\triangle ABC$ in plane $\mathcal{P}$, a point Q in space is defined such that $\angle AQB = \angle BQC = \angle CQA = 90°$. Point X is the point in plane $\mathcal{P}$ such that $\overline{QX}$ is perpendicular to plane $\mathcal{P}$. Given $\angle ABC = 40°$ and $\angle ACB = 75°$, find $\angle AXC$.

Credit This problem was proposed by Sandor Lehoczky.

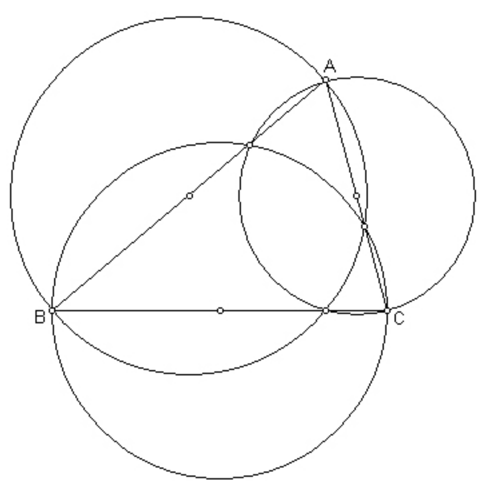
Comments The key insight in this problem is to realize that X is the orthocenter of triangle ABC. This, in turn, can be proven by showing that Q must lie on certain spheres. Once you identify that the point X is the orthocenter, the rest of the problem becomes an easy angle chase. *Solutions edited by Naoki Sato.*

Solution 1 by: Philip Shirey (12/PA)

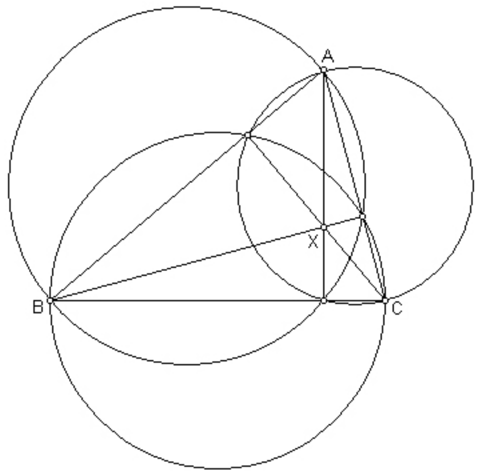
It is well known that in two dimensions, given points A and B, the locus of points P such that $\angle APB = 90°$ is the circle with diameter AB. (See diagram below.) The three-dimensional locus, such as in this problem, is a sphere. Thus, point Q is the intersection of three spheres.



We can better visualize the intersection of the three spheres by taking their intersection with the plane $\mathcal{P}$.



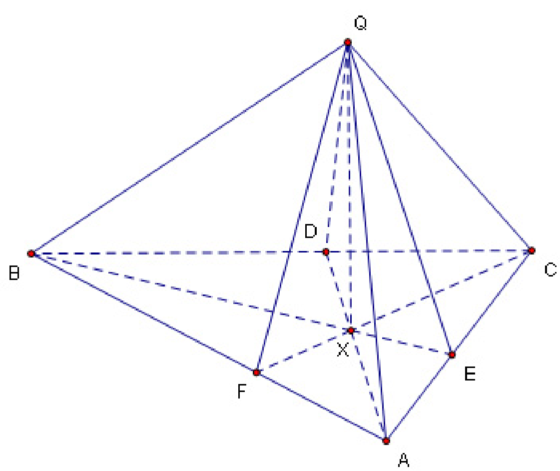
The triangle shown above is acute, so each circle interesects with the adjancent segments. As mentioned in the above theorem, any point on a circle will form a right triangle with its diameter. Because of this, these intersections form the altitudes of the triangle.



The intersection of two spheres is a circle. In the 2D representation of our 3D figure, these circles would be perpendicular to the plane of the page, which means they would be shown as segments in the 2D image. The endpoints of these segments are the interesections of the projected spheres (i.e., the circles). So, the altitudes represent the intersections of the spheres. As point X lies on plane $\mathcal{P}$, X is the intersection of the altitudes, otherwise known as the orthocenter.

Lastly, $\angle AXC = 140^\circ$ because the congruent angle on the opposite side of X forms a quadrilateral with two 90° angles with $\angle ABC$, which is 40°. So $\angle AXC = 180^\circ - 40^\circ = 140^\circ$.

Solution 2 by: Tan Zou (10/IN)



Let P_{ACQ} be the plane of $\triangle ACQ$. Draw a line through B and X such that it intersects side $\overline{AC}$ at point E. Draw EQ and let P_{BEQ} be the plane of $\triangle BEQ$. Then $\overline{BQ} \perp \overline{AQ}$ and

$\overline{BQ} \perp \overline{CQ}$, so $P_{BEQ} \perp P_{ACQ}$. Hence, $\overline{BE} \perp \overline{AC}$, because $\overline{BE}$ and $\overline{AC}$ are in P_{BEQ} and P_{ACQ}, respectively.

Similarly, if we draw $\overline{AD}$ and $\overline{CF}$ through point X, we can prove that they are perpendicular to $\overline{BC}$ and $\overline{AB}$, respectively. Therefore, X is the orthocenter of $\triangle ABC$.

We know that if H is the orthocenter of $\triangle ABC$, then $\angle A + \angle BHC = 180°$. To see this, let $\angle A = \theta$. Then $\angle ABH = 90° - \theta$, so $\angle BHF = \theta$, and $\angle BHC = 180° - \theta$.

Since X is the orthocenter and $\angle B = 40°$, $\angle AXC = 180° - 40° = 140°$.

Solution 3 by: Zhou Fan (12/NJ)

Let X be the origin of coordinate space, and let us use the notation $\vec{a}$ for the vector from point X to point A, etc.

Since QX is perpendicular to the plane containing ABC,

$$\vec{q} \cdot \vec{a} = \vec{q} \cdot \vec{b} = \vec{q} \cdot \vec{c} = 0.$$

Since $\angle AQB = \angle BQC = \angle CQA = 90°$,

$$(\vec{q} - \vec{a}) \cdot (\vec{q} - \vec{b}) = (\vec{q} - \vec{b}) \cdot (\vec{q} - \vec{c}) = (\vec{q} - \vec{c}) \cdot (\vec{q} - \vec{a}) = 0.$$

Expanding (by the distributive property of dot products), and substituting 0 for $\vec{q} \cdot \vec{a}$, $\vec{q} \cdot \vec{b}$, and $\vec{q} \cdot \vec{c}$ gives $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{c} = \vec{c} \cdot \vec{a} = -|\vec{q}|^2$. But if $\vec{a} \cdot \vec{b} = \vec{a} \cdot \vec{c}$, then $\vec{a} \cdot (\vec{b} - \vec{c}) = 0$, so AX is perpendicular to BC. Similarly, BX and CX are perpendicular to AC and AB, respectively, so X is the orthocenter of ABC.

Let CX intersect AB at C_1 and AX intersect BC at A_1; then $\triangle AC_1X \cong \triangle AA_1B$ because $\angle AC_1X = \angle AA_1B = 90°$. Therefore, $\angle AXC_1 = \angle ABA_1 = 40°$, and $\angle AXC = 140°$.