1/1/17. An increasing arithmetic sequence with infinitely many terms is determined as follows. A single die is thrown and the number that appears is taken as the first term. The die is thrown again and the second number that appears is taken as the common difference between each pair of consecutive terms. Determine with proof how many of the 36 possible sequences formed in this way contain at least one perfect square.

Credit This problem was taken from the book "St. Mary's College Mathematics Contest Problems."

Comments This is a straight-forward problem using modular arithmetic, requiring only some basic casework. We would like to point out that technically, the term "quadratic residue" only applies when the modulus is prime, and 0 is not included. For example, the quadratic residues modulo 7 are 1, 2, and 4. Otherwise, the term "square" modulo m should be used. *Solutions edited by Naoki Sato.*

Solution 1 by: Derrick Sund (12/WA)

Note: throughout this problem, I will use (a, b) to denote the infinite arithmetic sequence obtained from first rolling the number a, and then rolling the number b.

It is a well-known fact that if i is a quadratic residue (mod j), there are infinitely many perfect squares congruent to i (mod j), and that if k is not a quadratic residue (mod j), then there are no perfect squares congruent to k (mod j). Thus, if a is a quadratic residue (mod b), then the sequence (a, b) (which consists of all numbers greater than or equal to a which are congruent to a (mod b)) must contain a perfect square, and likewise, if a is not a quadratic residue (mod b), the sequence (a, b) cannot contain a perfect square.

Therefore, the sequence (a, b) will contain a perfect square if and only if a is a quadratic residue (mod b). Since it is also well-known that you can determine all quadratic residues (mod n) simply by squaring all numbers from 1 to n, inclusive, and finding their residues (mod n), we can finish the problem by finding the quadratic residues for mods 2, 3, 4, 5, and 6 ((mod 1) need not be considered, since $(a, 1)$ trivially contains all perfect squares greater than or equal to a).

The quadratic residues (mod 2) are 0 and 1. Therefore, (1,2), (2,2), (3,2), (4,2), (5,2), (6,2) all contain perfect squares.

The quadratic residues (mod 3) are 0 and 1. Therefore, (1,3), (3,3), (4,3), (6,3) all contain perfect squares, while (2,3), (5,3) do not.

The quadratic residues (mod 4) are 0 and 1. Therefore, (1,4), (4,4), (5,4) all contain perfect squares, while (2,4), (3,4), (6,4) do not.

The quadratic residues (mod 5) are 0, 1, and 4. Therefore, (1,5), (4,5), (5,5), (6,5) all contain perfect squares, while (2,5), (3,5) do not.

The quadratic residues (mod 6) are 0, 1, 3, and 4. Therefore, (1,6), (3,6), (4,6), (6,6) all contain perfect squares, while (2,6), (5,6) do not.

Thus, since 6 is the highest number that a die can roll, we have 27 sequences with perfect squares: (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,2), (2,2), (3,2), (4,2), (5,2), (6,2), (1,3), (3,3), (4,3), (6,3), (1,4), (4,4), (5,4), (1,5), (4,5), (5,5), (6,5), (1,6), (3,6), (4,6), (6,6).

Solution 2 by: Jeff Nanney (12/TX)

Denote the result of the first die toss d. Denote the result of the second die toss a. Naturally, $a, d \in \mathbb{N}$ such that $1 \le a, d \le 6$. We now seek to determine which ordered pairs (a, d) will yield at least one perfect square of the form $a(n-1)+d$, where $n \in \mathbb{N}$. Though a variety of approaches are available, the most natural is to examine the 6 cases according to the values of a. In particular, we will use the basic property that squaring all the members of a residue system yields each possible residue for a perfect square in that modulus. In general, we are seeking to find a solution in positive integers to the equation $an + d = x^2$, which is equivalent to finding for which d there exists some x such that $x^2 \equiv d \pmod{a}$.

1. Let $a = 1$. Thus, for $1 \le d \le 6$, we must find some integer x such that $x^2 \equiv d \pmod{1}$. Since all positive integers are congruent modulus 1, we know that all d are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, immediately noting that 9 is a perfect square attainable by all the sequences, regardless of the value of d. Thus, we have 6 sequences so far for which a perfect square is produced.

2. Let $a = 2$. For $1 \le d \le 6$, we must find some integer x such that $x^2 \equiv d \pmod{2}$. Because $0^2 = 0$ and $1^2 = 1$, and all members of the residue system are perfect squares, we know that all d are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, immediately noting that 9 is a perfect square attainable when d is odd, and 16 is a perfect square attainable when d is even. Thus, we have 6 more sequences for which a perfect square is produced.

3. Let $a = 3$. For $1 \le d \le 6$, we must find some integer x such that $x^2 \equiv d \pmod{3}$. Because $0^2 = 0$, $1^2 = 1$, and $2^2 \equiv 1 \pmod{3}$, we know that $d \equiv 0, 1 \pmod{3}$, or

$d = 1, 3, 4, 6$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 9 is a perfect square attainable when $d \equiv 0 \pmod 3$, and 16 is a perfect square attainable when $d \equiv 1 \pmod 3$. Thus, we have 4 more sequences for which a perfect square is produced.

4. Let $a = 4$. For $1 \le d \le 6$, we must find some integer x such that $x^2 \equiv d \pmod 4$. Because $0^2 = 0$, $1^2 = 1$, $2^2 \equiv 0 \pmod 4$, and $3^2 \equiv 1 \pmod 4$, we know that $d \equiv 0, 1 \pmod 4$, or $d = 1, 4, 5$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 16 is a perfect square attainable when $d \equiv 0 \pmod 4$, and 9 is a perfect square attainable when $d \equiv 1 \pmod 4$. Thus, we have 3 more sequences for which a perfect square is produced.

5. Let $a = 5$. For $1 \le d \le 6$, we must find some integer x such that $x^2 \equiv d \pmod 5$. Because $0^2 = 0$, $1^2 = 1$, $2^2 \equiv 4 \pmod 5$, $3^2 \equiv 4 \pmod 5$, and $4^2 \equiv 1 \pmod 5$, we know that $d \equiv 0, 1, 4 \pmod 5$, or $d = 1, 4, 5, 6$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 25 is a perfect square attainable when $d \equiv 0 \pmod 5$, 16 is a perfect square attainable when $d \equiv 1 \pmod 5$, and 4 is a perfect square attainable when $d \equiv 4 \pmod 5$. Thus, we have 4 more sequences for which a perfect square is produced.

6. Let $a = 6$. For $1 \le d \le 6$, we must find some integer x such that $x^2 \equiv d \pmod 6$. Because $0^2 = 0$, $1^2 = 1$, $2^2 \equiv 4 \pmod 6$, $3^2 \equiv 3 \pmod 6$, $4^2 \equiv 4 \pmod 6$, and $5^2 \equiv 1 \pmod 6$, we know that $d \equiv 0, 1, 3, 4 \pmod 5$, or $d = 1, 3, 4, 6$, are candidates to produce perfect squares in the sequence. To verify, we implement a simple check, noting that 25 is a perfect square attainable when $d \equiv 1 \pmod 6$, 9 is a perfect square attainable when $d \equiv 3 \pmod 6$, 16 is a perfect square attainable when $d \equiv 4 \pmod 6$, and 36 is a perfect square attainable when $d \equiv 0 \pmod 6$. Thus, we have 4 more sequences for which a perfect square is produced.

Combining the conclusions from each of the above 6 cases, we find that of the 36 possible sequences, exactly $6 + 6 + 4 + 3 + 4 + 4 = 27$ sequences contain at least one perfect square.

2/1/17. George has six ropes. He chooses two of the twelve loose ends at random (possibly from the same rope), and ties them together, leaving ten loose ends. He again chooses two loose ends at random and joins them, and so on, until there are no loose ends. Find, with proof, the expected value of the number of loops George ends up with.

Credit This problem, or some variation, is often used in interviews for quantitative positions on Wall Street.

Comments The easiest approach, as the solution by James Sundstrom illustrates, is to develop a recursive formula for the number of expected loops formed when starting with n ropes. However, it is also possible to find the expected value by counting all possible loops. This was done in a clever way by Scott Kovach. *Solutions edited by Naoki Sato.*

Solution 1 by: James Sundstrom (11/NJ)

Let E_n denote the expected value of the number of loops from this process starting with n ropes. Then we have the following lemma:

Lemma. For all natural numbers n, $E_n = E_{n-1} + \frac{1}{2n-1}$.

Proof. If the process starts with n ropes, after one loose end is selected there are $2n-1$ loose ends remaining, giving a probability of $\frac{1}{2n-1}$ that the other end of the same rope will be selected as the second choice. If this occurs, there is one loop already formed and $n-1$ loose ropes left, so the expected value for the number of loops formed is $1 + E_{n-1}$. There is a probability of $\frac{2n-2}{2n-1}$ that an end of a different rope will be chosen, leaving $n-1$ ropes (1 longer one and $n-2$ short ones). Then the expected value of the number of loops is E_{n-1}. Combining the two possibilities gives:

$$\begin{aligned} E_n &= \frac{1}{2n-1} \times (1 + E_{n-1}) + \frac{2n-2}{2n-1} \times E_{n-1} \\ &= \frac{1}{2n-1} + \left(\frac{1}{2n-1} + \frac{2n-2}{2n-1} \right) \times E_{n-1} \\ &= E_{n-1} + \frac{1}{2n-1}. \end{aligned}$$

$\square$

It is obvious that $E_1 = 1$, so we have:

$$E_2 = 1 + \frac{1}{3} = \frac{4}{3},$$
$$E_3 = \frac{4}{3} + \frac{1}{5} = \frac{23}{15},$$
$$E_4 = \frac{23}{15} + \frac{1}{7} = \frac{176}{105},$$
$$E_5 = \frac{176}{105} + \frac{1}{9} = \frac{563}{315},$$
$$E_6 = \frac{563}{315} + \frac{1}{11} = \frac{6508}{3465}.$$

Therefore, if George has six ropes which he ties together by randomly selecting two loose ends at a time to tie together, the expected value of the number of loops he will end with is $\frac{6508}{3465}$.

Solution 2 by: Scott Kovach (10/TN)

First consider the number of different ways to tie n ropes together. The first tie can be done in $\binom{2n}{2}$ ways, leaving $2n-2$ loose ends. The next can be done in $\binom{2n-2}{2}$ ways, the next in $\binom{2n-4}{2}$, and so on. The order that the ties are made doesn't matter, however, so we must divide the product of these binomial coefficients by $n!$. The number of ways therefore is

$$f(n) = \frac{\binom{2n}{2}\binom{2n-2}{2}\cdots\binom{2}{2}}{n!} = \frac{\frac{(2n)!}{2(2n-2)!}\cdot\frac{(2n-2)!}{2(2n-4)!}\cdots\frac{2!}{2(0)!}}{n!} = \frac{(2n)!}{2^n n!}.$$

Now consider the number of ways to tie n ropes together to make a loop. The ropes can be sequentially tied to each other in any order, so there are $(n-1)!$ ways to order them. The first end of the first rope can be tied to either end of the second, the remaining end of the second to either end of the third, and so on. There are 2^{n-1} ways to do this, so the total number of ways to tie the n ropes together into a single loop is $g(n) = 2^{n-1}(n-1)!$.

Finally, we count the total number of loops among all the possible tyings. There are $\binom{6}{1}$ ways to choose one rope and $g(1)$ ways to tie it into one loop, and $f(5)$ ways to tie the remaining ropes together. Similarly, there are $\binom{6}{2}$ ways to choose two ropes, $g(2)$ ways to tie them into one loop, and $f(4)$ ways to tie the other four ropes together. Extending this process counts every possible loop of any size. The total number of loops is

$$\sum_{k=1}^{6}\binom{6}{k}g(k)f(6-k) = 19524.$$

There are $f(6) = 10395$ tyings, so the expected value is

$$\frac{19524}{10395} = \frac{6508}{3465}.$$

3/1/17. Let r be a nonzero real number. The values of z which satisfy the equation

$$r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) = 0$$

are plotted on the complex plane (i.e. using the real part of each root as the x-coordinate and the imaginary part as the y-coordinate). Show that the area of the convex quadrilateral with these points as vertices is independent of r, and find this area.

Credit This problem was proposed by Dave Patrick of AoPS and Erin Schram of the NSA.

Comments Many students seem to have been intimidated by the complicated looking quartic equation and the setting of the complex plane. The first step is to find the factors of the quartic. This is really the bulk of the problem, and was accomplished with a variety of approaches, as the following solutions illustrate. The next step is to plot the roots in the complex plane, which are found to form a trapezoid. Some students merely plugged the quartic into software such as *Mathematica*, but you still need to show justification that the roots so produced are in fact correct. *Solutions edited by Naoki Sato.*

Solution 1 by: Daniel Jiang (11/IN)

The constant coefficient of the equation can be factored so that the equation becomes:

$$r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^4 + 1)(r^4 + 1) = 0.$$

Factoring would lead us to the roots of the equation as a function of r. From what we have so far, we can see that the factors of the equation may look like

$$[r^2z^2 + \cdots + (r^4 + 1)][r^2z^2 + \cdots + (9r^4 + 1)].$$

The given equation has powers of z^4, z^2, and z, so at this stage, we let the factors take the form

$$[r^2z^2 + az + (r^4 + 1)][r^2z^2 + bz + (9r^4 + 1)].$$

Expanding, we get:

$$r^4z^4 + (ar^2 + br^2)z^3 + (10r^6 + 2r^2 + ab)z^2 + (a + b + 9ar^4 + br^4)z + (9r^8 + 10r^4 + 1) = 0.$$

Comparing this to the given equation, we know that there is no z^3 term, so $a + b = 0$, and from the z^2 term, we see that $ab = -4r^2$. Using these, it is easy to see that $(a, b) = (-2r, 2r)$ or $(2r, -2r)$, but then testing in the z term, we see that a must be $-2r$, so $(a, b) = (-2r, 2r)$.

The factored form of the equation is then

$$(r^2z^2 - 2rz + r^4 + 1)(r^2z^2 + 2rz + 9r^4 + 1) = 0.$$

Now the quadratic formula can be applied to each of the quadratic factors, which gives us the four roots as: $\frac{1}{r} \pm ri$ and $-\frac{1}{r} \pm 3ri$.

Plotting the points $\left(\frac{1}{r}, r\right), \left(\frac{1}{r}, -r\right), \left(-\frac{1}{r}, 3r\right), \left(-\frac{1}{r}, -3r\right)$, we have a trapezoid with two bases of length $|6r|$ and $|2r|$ and a height of $\frac{2}{|r|}$. The area of a trapezoid is $\frac{b_1+b_2}{2} \cdot h$, so the area is $\frac{|8r|}{2} \cdot \frac{2}{|r|} = 8$. The r cancels out; thus, the area of this convex quadrilateral is 8 independent of r.

Solution 2 by: Joshua Horowitz (11/CT)

The given expression can be written and factored as the difference of two squares:

$$\begin{aligned} r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) = 0 \\ \Rightarrow\ [r^2z^2 + (5r^4 + 1)]^2 - (2rz + 4r^4)^2 = 0 \\ \Rightarrow\ [r^2z^2 + 2rz + (9r^4 + 1)][r^2z^2 - 2rz + (r^4 + 1)] = 0. \end{aligned}$$

So the roots of the original equation are the roots of $P(z) = r^2z^2 + 2rz + (9r^4 + 1)$ combined with the roots of $Q(z) = r^2z^2 - 2rz + (r^4 + 1)$. Each of these is a quadratic with real coefficients. The quadratic P has discriminant $-36r^6$ and the quadratic Q has discriminant $-4r^6$. Both of these are negative (since r is a nonzero real) so the roots of P and the roots of Q form conjugate pairs.

Using the quadratic formula we can compute a root $p = -\frac{1}{r} + 3ri$ of P and a root $q = \frac{1}{r} + ri$ of Q. These two roots and their conjugates will form an isosceles trapezoid with the real axis as an axis of symmetry. The area of this trapezoid (the desired answer to this problem) will be the height times the sum of half the bases:

$$|\operatorname{Re} p - \operatorname{Re} q|\,(|\operatorname{Im} p| + |\operatorname{Im} q|) = \left|\frac{2}{r}\right|(|3r| + |r|) = \left|\frac{2}{r}\right||4r| = 8,$$

where $\operatorname{Re} z$ and $\operatorname{Im} z$ denote the real and imaginary parts of the complex number z, respectively.

Solution 3 by: Linda Liu (11/GA)

We have that

$$\begin{aligned} r^4z^4 + (10r^6 - 2r^2)z^2 - 16r^5z + (9r^8 + 10r^4 + 1) = 0 \\ \Rightarrow\ r^4z^4 + (6r^6 - 2r^2)z^2 + 9r^8 - 6r^4 + 1 + 4r^6z^2 - 16r^5z + 16r^4 = 0 \\ \Rightarrow\ r^4z^4 + 2(3r^4 - 1)r^2z^2 + (3r^4 - 1)^2 + (2r^3z - 4r^2)^2 = 0 \\ \Rightarrow\ (r^2z^2 + 3r^4 - 1)^2 + (2r^3z - 4r^2)^2 = 0, \end{aligned}$$

so

$$(r^2z^2 + 3r^4 - 1)^2 = -(2r^3z - 4r^2)^2.$$

Taking the square root of both sides gives the two equations

$$r^2z^2 + 3r^4 - 1 = (2r^3z - 4r^2)i \;\Rightarrow\; r^2z^2 - 2r^3zi + 3r^4 + 4r^2i - 1 = 0,$$

and

$$r^2z^2 + 3r^4 - 1 = -(2r^3z - 4r^2)i \;\Rightarrow\; r^2z^2 + 2r^3zi + 3r^4 - 4r^2i - 1 = 0.$$

Applying the quadratic formula to the first quadratic equation produces the roots

$$-\frac{1}{r} + 3ri \quad \text{and} \quad \frac{1}{r} - ri.$$

Applying the quadratic formula to the second quadratic equation produces the roots

$$-\frac{1}{r} - 3ri \quad \text{and} \quad \frac{1}{r} + ri.$$

These four complex numbers then form a trapezoid with height $2/|r|$ and bases $|2r|$ and $|6r|$, so the area of the trapezoid is

$$\frac{|2r| + |6r|}{2} \times \frac{2}{r} = 8.$$

4/1/17. Homer gives mathematicians Patty and Selma each a different integer, not known to the other or to you. Homer tells them, within each other's hearing, that the number given to Patty is the product ab of the positive integers a and b, and that the number given to Selma is the sum $a+b$ of the same numbers a and b, where $b > a > 1$. He doesn't, however, tell Patty or Selma the numbers a and b. The following (honest) conversation then takes place:

Patty: "I can't tell what numbers a and b are."
Selma: "I knew before that you couldn't tell."
Patty: "In that case, I now know what a and b are."
Selma: "Now I also know what a and b are."

Supposing that Homer tells *you* (but neither Patty nor Selma) that neither a nor b is greater than 20, find a and b, and prove your answer can result in the conversation above.

Credit This problem comes from the Carnegie Mellon Math Studies Problem Seminar.

Comments What makes this problem tricky is that it's not just a problem: It's a problem inside a problem. It requires you to place yourselves in the shoes of Patty and Selma, and not only make the same deductions they make, but deduce which conditions could have led to those deductions. Some careful casework leads to the answer. Note that in the following solutions, Meir Lakhovsky shows that $(a, b) = (4, 13)$ is a viable solution, and Jeffrey Manning shows that it is the only solution. *Solutions edited by Naoki Sato.*

Solution 1 by: Meir Lakhovsky (10/WA)

Some trial and error leads us to $a = 4$, $b = 13$. Let us show that the conversation mentioned could take place. Patty was given the number 52 and Selma was given the number 17.

Patty said, "I can't tell what numbers a and b are," which is true since the product 52 could be achieved through either the numbers (2,26) or (4,13).

Selma answers "I knew before that you couldn't tell," which is also true, since for each possible pair (a, b) that adds up to 17, namely (2,15), (3,14), (4,13), (5,12), (6,11), (7,10), and (8,9), there are at least two possible solutions that give rise to the product ab.

Patty then says, "In that case, I now know what a and b are," which is true because if (a, b) was equal to (2,26), then Selma would have had the number 28, which means she could not have made her former statement because (a, b) could have been (5,23) in which case Patty would have been able to figure out what a and b are before any statements were made. Since Patty knows the product is 52 and (a, b) is not (2,26), she knows that (a, b) is (4,13), the only other option.

Selma now says, "Now I also know what a and b are," which is true since the only pair from (2,15), (3,14), (4,13), (5,12), (6,11), (7,10), and (8,9) in which Patty could have made her later statement is (4,13) (reasoning shown above). Let us show why (2,15), (3,14), (5,12), (6,11), (7,10), and (8,9) don't work. If (2,15) were the numbers, then Patty would have had the number 30 and would not have been able to make her later statement since both (2,15) and (5,6) yield sums for which Selma would have been able to make her former statement. Likewise, for (3,14), Patty would not have been able to make her statement since both (3,14) and (2,21) yield sums for which Selma would have been able to make her former statement. By the same logic, the pair for (5,12) is (3,20); the pair for (6,11) is (2,33); the pair for (7,10) is (2,35); and the pair for (8,9) is (3,24).

Thus, $(a, b) = (4, 13)$ could have resulted in the described conversation.

Solution 2 by: Jeffrey Manning (10/CA)

Notice that Patty could tell what a and b were if and only if there is exactly one way to factor ab into the product of two distinct integers, both greater than one (for the rest of the solution we will use the word factorization to mean factorization into two distinct factors both greater than 1), in that case a and b are the two factors. This would only happen when $ab = pq$ where p and q are primes, in which case a and b would be p and q or when $ab = p^3$ where p is prime, in which case $a = p^2$ and $b = p$.

For Selma to already know that Patty couldn't tell what a and b were, it must be impossible to write $a + b$ as the sum of two distinct primes or as the sum of a prime and its square. Since $a + b \leq 40$, the possible values for $a + b$ are 11, 17, 23, 27, 29, 35, and 37 (notice that we do need to consider primes and squares of primes greater than 20 because *Selma* doesn't know that $a, b \leq 20$).

For the third line of the conversation to be true, there must be exactly one factorization of ab such that the sum of the factors cannot be written as the sum of two distinct primes or as the sum of a prime and its square. If $a, b > 1$, then $(a - 1)(b - 1) = ab - (a + b) + 1 > 0$, so $a + b < ab + 1$, so Patty knows that $a + b < 401$. This means Patty knows that $a + b$ is either odd or twice a prime (Goldbach's conjecture states that any even integer ≥ 4 is the sum of two primes, not necessarily distinct. Although this has not been proven, it has been verified for all values we are concerned with). So we only need to consider factorizations of ab such that the sum of the factors is not divisible by 4 (we can't have $a + b = 2(2) = 4$ because that would mean ab is 3 or 4). Notice that if $ab = 4p$ where p is an odd prime, then the only possible values of a and b are 4 and p.

For the fourth line to be true there must be only one way to write $a + b$ as the sum of two numbers (> 1) whose product satisfies these conditions.

We have:

If $a+b=11=4+7=2+9$, then we could have $ab=28$ or 18.
If $a+b=23=4+19=7+16$, then we could have $ab=76$ or 112.
If $a+b=27=4+23=2+25$, then we could have $ab=92$ or 50.
If $a+b=29=13+16=2+27$, then we could have $ab=208$ or 54.
If $a+b=35=4+31=32+3$, then we could have $ab=124$ or 96.
If $a+b=37=8+29=32+5$, then we could have $ab=232$ or 160.

We will show that each of these possible values of ab satisfy the conditions for the third line to be true. The numbers 28, 76, 92 and 124 are all in the form $4p$ so they all work.

If $ab=18=2\cdot3\cdot3$ then the only factorization, other than $2\cdot9$, is $3\cdot6$ which gives $a+b=9=2+7$.

If $ab=50=2\cdot5\cdot5$ then the only factorization, other than $2\cdot25$, is $5\cdot10$ which gives $a+b=15=2+13$.

If $ab=54=2\cdot3\cdot3\cdot3$ then the only factorizations, other than $2\cdot27$, are $6\cdot9$ which gives $a+b=15=2+13$, and $3\cdot18$ which gives $a+b=21=2+19$.

If $ab=112=2\cdot2\cdot2\cdot2\cdot7$ then the only factorizations other than $7\cdot16$ such that $4\nmid a+b$, are $8\cdot14$ which gives $a+b=22=5+17$, and $2\cdot56$ which gives $a+b=58=5+53$.

If $ab=208=2\cdot2\cdot2\cdot2\cdot13$ then the only factorizations other than $13\cdot16$ such that $4\nmid a+b$, are $8\cdot26$ which gives $a+b=34=3+31$, and $2\cdot104$ which gives $a+b=106=47+59$.

If $ab=96=2\cdot2\cdot2\cdot2\cdot2\cdot3$ then the only factorizations other than $3\cdot32$ such that $4\nmid a+b$, are $6\cdot16$ which gives $a+b=22=5+17$, and $2\cdot48$ which gives $a+b=50=19+31$.

If $ab=160=2\cdot2\cdot2\cdot2\cdot2\cdot5$ then the only factorizations other than $5\cdot32$ such that $4\nmid a+b$, are $10\cdot16$ which gives $a+b=26=3+23$, and $2\cdot80$ which gives $a+b=82=29+53$.

If $ab=232=2\cdot2\cdot2\cdot29$ then the only factorizations, other than $8\cdot29$, are $4\cdot58$ which gives $a+b=62=3+59$, and $2\cdot116$ which gives $a+b=118=5+113$.

Therefore $a+b=11$, 23, 27, 29, 35, and 37 don't satisfy the fourth line, so $a+b=17$.

Thus the possible ordered pairs (a,b) are:

$(2,15)\Rightarrow ab=30$, but this can be factored as $5\cdot6$ and $5+6=11$.
$(3,14)\Rightarrow ab=42$, but this can be factored as $2\cdot21$ and $2+21=23$.
$(4,13)\Rightarrow ab=52$, which can be written in the form $4p$ so it satisfies the third line.
$(5,12)\Rightarrow ab=60$, but this can be factored as $3\cdot20$ and $3+20=23$.
$(6,11)\Rightarrow ab=66$, but this can be factored as $2\cdot33$ and $2+33=35$.
$(7,10)\Rightarrow ab=70$, but this can be factored as $2\cdot35$ and $2+35=37$.
$(8,9)\Rightarrow ab=72$, but this can be factored as $3\cdot24$ and $3+24=27$.

Thus the only value of ab that satisfies the third line is 52, so $a + b = 17$ satisfies the fourth line. Thus the only possible value of (a, b) is (4,13).

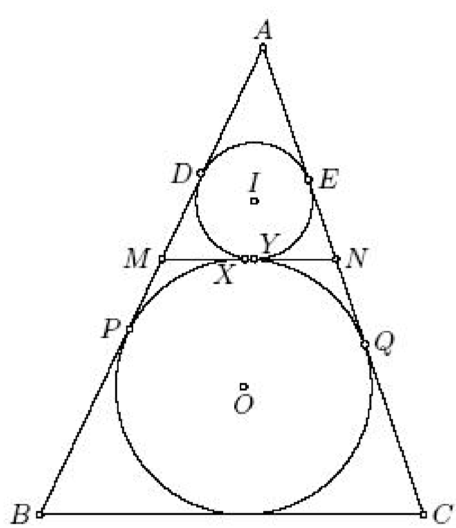
5/1/17. Given triangle ABC, let M be the midpoint of side $\overline{AB}$ and N be the midpoint of side $\overline{AC}$. A circle is inscribed inside quadrilateral $NMBC$, tangent to all four sides, and that circle touches $\overline{MN}$ at point X. The circle inscribed in triangle AMN touches $\overline{MN}$ at point Y, with Y between X and N. If $XY = 1$ and $BC = 12$, find, with proof, the lengths of the sides $\overline{AB}$ and $\overline{AC}$.

Credit This problem was proposed by Richard Rusczyk.

Comments This geometry problem is best solved using a "side chase" (as opposed to an "angle chase"), in which the relations between lengths of line segments are written down until there are a sufficient number of them that they can be solved algebraically. Any approach of this kind will almost inevitably lead to the answer. But this is not the only possible approach, as the following solutions will show. We recommend that when submitting a solution to a geometry problem to also provide a diagram, so that the grader does not have to draw one. *Solutions edited by Naoki Sato.*

Solution 1 by: Tony Liu (11/IL)

Let AB and AC be tangent to the incircle of $\triangle AMN$ at D and E. Similarly, let AB and AC be tangent to the circle inscribed in $MNCB$ at P and Q. Let $AC = 2b$, $AB = 2c$. Note that $MN = \frac{1}{2} \cdot BC = 6$, and let $s = \frac{1}{2}(b + c + 6)$ be the semiperimeter of $\triangle AMN$.



Because $MNCB$ has an inscribed circle, $MN + BC = MB + NC$, or $b + c = 18$. By equal tangents around the incircle of $\triangle AMN$, it is easy to see that $MY = s - b$ and $NY = s - c$.

For example,

$$\begin{aligned}
2 \cdot MY &= MY + MD \\
&= (AM - AD) + (MN - YN) \\
&= AM - AE + MN - EN \\
&= AM + MN - AN \\
&= AM + MN + AN - 2AN \\
&= 2s - 2b \\
\Rightarrow MY &= s - b.
\end{aligned}$$

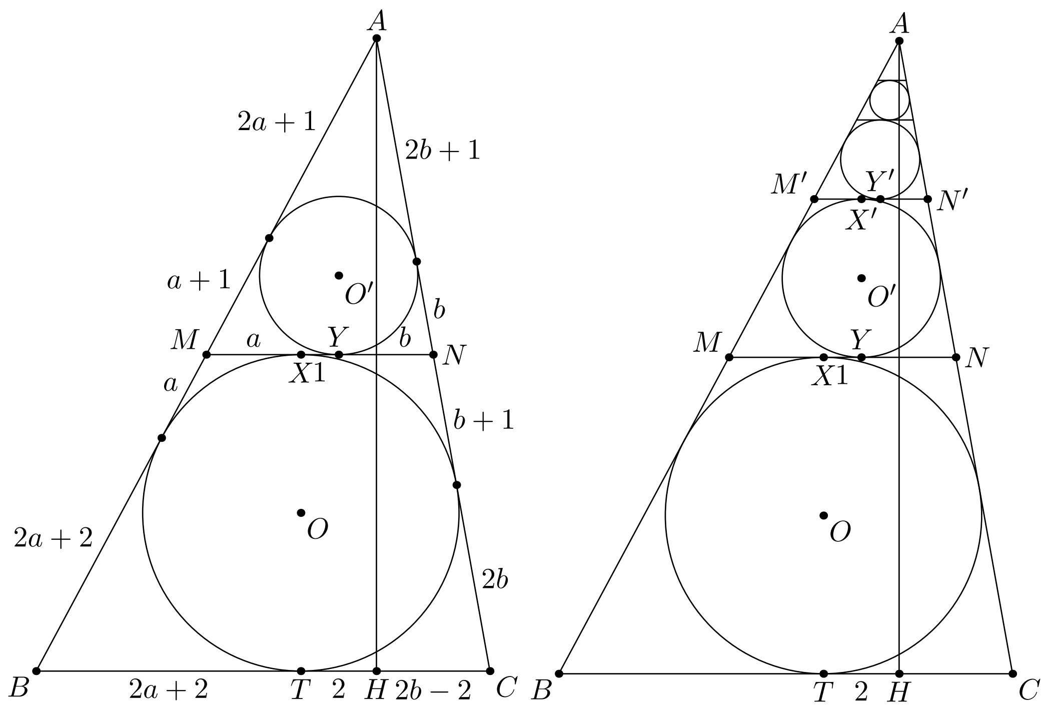
Moreover, the circle inscribed in $MNCB$ is the excircle of $\triangle AMN$, opposite of A. This implies that $NX = MY$. Indeed, we have

$$\begin{aligned}
2 \cdot AP &= AP + AQ \\
&= (AM + MX) + (AN + NX) \\
&= 2s \\
\Rightarrow AP = AQ &= s.
\end{aligned}$$

Thus, $MY = s - b = AQ - AN = NQ = NX$. From this, we have $1 = XY = NX - NY = (s-b)-(s-c) = c-b$. Combined with $b+c = 18$, we have $AB = 2c = 19$ and $AC = 2b = 17$.

Solution 2 by: Alan Deng (12/NY)

Let $MX = a$ and $NY = b$. By properties of tangents to a circle and the fact that triangle AMN is similar to triangle ABC with a ratio of similitude of 1:2 (since M is the midpoint of AB and N is the midpoint of AC), we can label the following diagram on the left, except for the length of TH, where H is the foot of the altitude from A to BC.



The ratio of the radius of circle O to that of O' is 2:1, since they are inscribed in similar triangles of that ratio. The distance XY is 1. Draw a line $M'N'$ parallel to MN that is also tangent to circle O', as shown in the diagram on the right. We also have points X' and Y' on $M'N'$ that correspond to points X and Y on MN. In fact, X' and Y' are the images of X and Y, respectively, of the similitude through A by ratio 1:2, so $X'Y' = XY/2 = 1/2$. Furthermore, $X'Y$ is parallel to AH since $M'N'$ and MN are parallel tangents to circle O'.

If we continue this process of drawing tangents and circles towards point A, we obtain line segments of length 1, 1/2, 1/4, 1/8, and so on, and the union of their orthogonal projections onto BC is TH. Hence,

$$TH = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2.$$

Now, we have two right triangles ABH and ACH that share the same height. Then

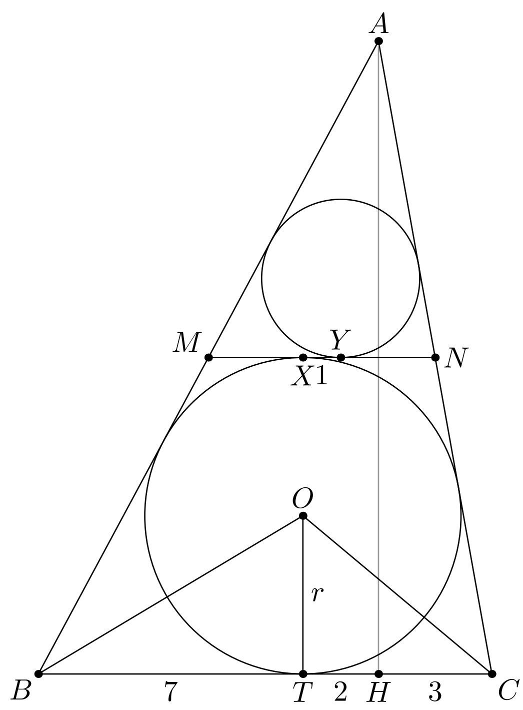
$$\begin{aligned} AB^2 - BH^2 &= AH^2 = AC^2 - CH^2 \\ \Rightarrow\ (6a+4)^2 - (2a+4)^2 &= (6b+2)^2 - (2b-2)^2 \\ \Rightarrow\ 32a^2 + 32a &= 32b^2 + 32b \\ \Rightarrow\ 4a^2 + 4a &= 4b^2 + 4b \\ \Rightarrow\ 4a^2 + 4a + 1 &= 4b^2 + 4b + 1 \\ \Rightarrow\ (2a+1)^2 &= (2b+1)^2, \end{aligned}$$

so $a = b$. Since $BC = 2a + 2b + 2 = 12$, $a = b = 5/2$. Hence, $AB = 6a + 4 = 19$ and $AC = 6b + 2 = 17$.

Solution 3: Based on the solution by Logan Daum (11/AK)

By the side chasing argument in Solution 1, we get that $MX = NY$. (For example, this follows immediately from $MY = NX$.) Since $XY = 1$ and $MN = 6$, we have that $MX = NY = 5/2$, and we can argue that $TH = 2$ as in Solution 2. (Actually, Logan argues this in a more direct way than Alan. Let Z be the intersection of AH and MN. What do you notice about triangles AZY and TXY?)

Now, T is the image of Y under the homothety through A by a factor of 2, so $CT = 2NY = 5$, which means $BT = BC - CT = 12 - 5 = 7$. Also, $CH = CT - TH = 3$ and $BH = BC - CH = 9$.



Let r be the inradius of triangle ABC, so the height of trapezoid $BMNC$ is $2r$, so then the height of triangle ABC is $AH = 4r$. Then

$$\tan \frac{B}{2} = \frac{OT}{BT} = \frac{r}{7},$$

and

$$\tan B = \frac{AH}{BH} = \frac{4r}{9}.$$

By the double angle formula,

$$\tan B = \frac{2\tan(B/2)}{1-\tan^2(B/2)}.$$

Substituting, we obtain

$$\begin{aligned} \frac{4r}{9} &= \frac{2r/7}{1-(r/7)^2} = \frac{14r}{49-r^2} \\ \Rightarrow\ 196-4r^2 &= 126 \\ \Rightarrow\ r^2 &= \frac{70}{4} = \frac{35}{2} \\ \Rightarrow\ r &= \sqrt{\frac{35}{2}} \\ \Rightarrow \tan B &= \frac{4r}{9} = \frac{4}{9}\sqrt{\frac{35}{2}} = \frac{2\sqrt{70}}{9}. \end{aligned}$$

Since $\tan B$ is positive, B is acute. Also,

$$\begin{aligned} \cos^2 B &= \frac{\cos^2 B}{\cos^2 B + \sin^2 B} \\ &= \frac{1}{1+\frac{\sin^2 B}{\cos^2 B}} \\ &= \frac{1}{1+\tan^2 B} \\ &= \frac{1}{1+\frac{4\cdot 70}{81}} \\ &= \frac{81}{361}, \end{aligned}$$

so

$$\cos B = \sqrt{\frac{81}{361}} = \frac{9}{19}.$$

Hence,

$$AB = \frac{BH}{\cos B} = \frac{9}{9/19} = 19.$$

Similarly,

$$\tan C = \frac{AH}{CH} = \frac{4r}{3} = \frac{4}{3}\sqrt{\frac{35}{2}} = \frac{2\sqrt{70}}{3},$$

so C is also acute, and

$$\begin{aligned}\cos^2 C &= \frac{1}{1+\tan^2 C}\\ &= \frac{1}{1+\frac{4\cdot 70}{9}}\\ &= \frac{9}{289},\end{aligned}$$

so

$$\cos C = \sqrt{\frac{9}{289}} = \frac{3}{17}.$$

Hence,

$$AC = \frac{AH}{\cos C} = \frac{3}{3/17} = 17.$$