USA Mathematical Talent Search

Solutions to Problem 1/4/16

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1/4/16. Determine with proof the number of positive integers n such that a convex regular polygon with n sides has interior angles whose measures, in degrees, are integers.

Credit We are grateful to Professor Gregory Galperin, one of the world's most powerful problem posers, for suggesting this problem for the USAMTS program.

Comments Most students took the straightforward approach illustrated by Jake Snell and Kim Scott below. A few looked at the exterior angles instead of the interior angles, as shown by Zachary Abel. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Jake Snell (11/NJ)

We know that the sum of interior angles in any n-gon is $180° \times (n-2)$. Therefore, since we are considering only regular polygons, each interior angle is congruent and now

$$m(\text{each interior angle}) = \frac{180°(n-2)}{n} = \frac{180°n - 360°}{n} = 180° - \frac{360°}{n}$$

Each interior angle will have an integer measure in degrees only if $\frac{360}{n}$ is an integer. Thus, n must be a factor of 360. We can construct these factors since we know that $360 = 2^3 \cdot 3^2 \cdot 5$. We seek all nonnegative integers a, b and c such that $2^a \cdot 3^b \cdot 5^c$ divides $2^3 \cdot 3^2 \cdot 5$. Since 2, 3, and 5 are all primes, $2^a | 2^3$, $3^b | 3^2$, and $5^c | 5$. $2^a | 2^3$ implies $\frac{2^3}{2^a} = 2^{(3-a)}$ is an integer. Therefore, $a \in \{0, 1, 2, 3\}$. In a similar manner, we find that $b \in \{0, 1, 2\}$ and $c \in \{0, 1\}$. Since $2^a \cdot 3^b \cdot 5^c$ is the prime factorization of a nonnegative integer, and since no two nonnegative numbers have the same prime factorization, given a unique combination of a, b and c, $2^a \cdot 3^b \cdot 5^c$ is a unique nonnegative integer. The number of unique combinations of a, b and c is simply the product of the number of different values they could be, or $4 \cdot 3 \cdot 2 = 24$. However, we must subtract 2 from this result since $1 = 2^0 \times 3^0 \cdot 5^0$ and $2 = 2^1 \cdot 3^0 \cdot 5^0$ do not yield polygons. Our answer is $n = 24 - 2 = 22$. //

Solution 2 by: Kim Scott (10/MA)
Since any polygon with $n \geq 3$ sides can be divided into $n-2$ triangles (each with total angle measure $180°$), the total degree measure of its interior angles is $180(n-2)°$. In a regular polygon, every interior angle has the same degree measure, which must thus be

$$\frac{180(n-2)}{n} = 180 - \frac{360}{n}$$

This is an integer exactly when $\frac{360}{n}$ is an integer, which is true iff n is a factor of 360.

Since $360 = 2^3 \times 3^2 \times 5$, its factors are of the form $2^a \times 3^b \times 5^c$, with $0 \leq a \leq 3$, $0 \leq b \leq 2$, and $0 \leq c \leq 1$. Since there are 4 values for a, 3 for b, and 2 for c, 360 has $4 * 3 * 2 = 24$ integral factors, and $\frac{360}{n}$ is an integer for 24 values of n. However, this count includes $n = 1$ and $n = 2$, which do not correspond to valid values for the number of sides of a regular polygon.

Therefore, there are $24 - 2 = 22$ positive integers n such that a convex regular polygon with n sides has interior angles whose measures, in degrees, are integers.

Solution 3 by: Zachary Abel (11/TX)
The interior angle is an integer if and only if the exterior angle is an integer because these two angles add to $180°$. Since the exterior angle measures $360°/n$, the condition holds if and only if n is a divisor of 360. Since $360 = 2^3 \cdot 3^2 \cdot 5$, this number has $4 \cdot 3 \cdot 2 = 24$ factors. But n must be at least 3, so we reject the possibilities that $n = 1$ or $n = 2$ and conclude that n may equal any of the other **22** divisors of 360.

2/4/16. Find positive integers a, b, and c such that

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}}.$$

Prove that your solution is correct. (Warning: numerical approximations of the values do not constitute a proof.)

Credit This is based on Problem 80 on page 42 of *Problems from the History of Mathematics*, by Lévárdi and Sain, a book published in Hungarian in Budapest, 1982. Problem 80 was attributed to the Indian mathematician Bhaskara (1114 - ca. 1185).

Comments Most students squared both sides of the given equation to get the solution. Examples are given below by Jason Ferguson and Tony Liu. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Jason Ferguson (12/TX)

If the ordered triple of real numbers (a, b, c) satisfy the problem condition, then so will any permutation (a', b', c') of (a, b, c). Thus, we may assume without loss of generality that $a \leq b \leq c$.

Upon squaring both sides of the equation

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}},$$

we obtain

$$\begin{aligned} a+b+c+2\sqrt{ab}+2\sqrt{ac}+2\sqrt{bc} &= 219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280} \\ &= 219+12\sqrt{70}+30\sqrt{14}+84\sqrt{5}. \end{aligned}$$

As a, b, and c are integers with $a \leq b \leq c$, it can be the case that $a+b+c=219$, $2\sqrt{ab}=12\sqrt{70}$, $2\sqrt{ac}=30\sqrt{14}$, and $2\sqrt{bc}=84\sqrt{5}$. Then

$$\sqrt{ab} = 6\sqrt{70}, \tag{1}$$
$$\sqrt{ac} = 15\sqrt{14}, \tag{2}$$
$$\sqrt{bc} = 42\sqrt{5}. \tag{3}$$

Multiplying (1), (2), and (3) gives

$$abc = 264600, \tag{4}$$

and squaring (1), (2), and (3) gives

$$ab = 2520, \tag{5}$$
$$ac = 3150, \tag{6}$$
$$bc = 8820, \tag{7}$$

respectively. Dividing (4) by (7), (6), and (5), respectively give $a = 30$, $b = 84$, and $c = 105$. Then indeed $a + b + c = 219$, so $\boxed{(a, b, c) = (30, 84, 105)}$ satisfy

$$\sqrt{a} + \sqrt{b} + \sqrt{c} = \sqrt{219 + \sqrt{10080} + \sqrt{12600} + \sqrt{35280}},$$

as desired. QED

Solution 2 by: Tony Liu (10/IL)
Squaring the given equation, we obtain

$$a + b + c + 2\sqrt{ab} + 2\sqrt{bc} + 2\sqrt{ca} = 219 + \sqrt{10080} + \sqrt{12600} + \sqrt{35280}.$$

Since there are three radical terms on the right side (which are not integers), the three radicals on the left side can match up correspondingly. Also note that because a, b, c are positive integers, this implies $a + b + c = 219$. Without loss of generality, we may assume $b \le a \le c \Rightarrow ab \le bc \le ca$ to obtain the following system of equations:

$$\begin{aligned}
\sqrt{ab} = \frac{1}{2}\sqrt{10080} &= 6\sqrt{70} \\
\sqrt{bc} = \frac{1}{2}\sqrt{12600} &= 15\sqrt{14} \\
\sqrt{ca} = \frac{1}{2}\sqrt{35280} &= 42\sqrt{5}
\end{aligned}$$

Multiplying the three gives $abc = 264600 \Rightarrow \sqrt{abc} = 210\sqrt{6}$. Thus, we can solve for a, b, c:

$$\sqrt{a} = \frac{\sqrt{abc}}{\sqrt{bc}} = \frac{210\sqrt{6}}{15\sqrt{14}} = 2\sqrt{21} \Rightarrow a = 84.$$

$$\sqrt{b} = \frac{\sqrt{abc}}{\sqrt{ca}} = \frac{210\sqrt{6}}{42\sqrt{5}} = \sqrt{30} \Rightarrow b = 30.$$

$$\sqrt{c} = \frac{\sqrt{abc}}{\sqrt{ab}} = \frac{210\sqrt{6}}{6\sqrt{70}} = \sqrt{105} \Rightarrow c = 105.$$

Checking, we note that $a + b + c = 84 + 30 + 105 = 219$ still holds. Finally, we conclude that $a = 84, b = 30, c = 105$ satisfy the given equation.

3/4/16. Find, with proof, a polynomial $f(x, y, z)$ in three variables, with integer coefficients, such that for all integers a, b, c, the sign of $f(a, b, c)$ (that is, positive, negative, or zero) is the same as the sign of $a + b\sqrt[3]{2} + c\sqrt[3]{4}$.

Credit This problem was devised by Dr. Erin Schram of the National Security Agency.

Comments Most students used algebraic manipulation to arrive at a solution; Tony Liu and Laura Starkston give example solutions.

Solutions edited by Richard Rusczyk.

Solution 1 by: Tony Liu (10/IL)

We claim that the polynomial $f(a, b, c) = a^3 + 2b^3 + 4c^3 - 6abc$ has the desired properties. Our proof begins with the following lemma:

Lemma: *The expressions $s = p^3 + q^3 + r^3 - 3pqr$ and $t = p + q + r$ have the same sign for real numbers p, q, r that are not all equal.*

Proof: We note the following identity:

$$\begin{aligned} p^3 + q^3 + r^3 - 3pqr &= (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) \\ &= \frac{1}{2}(p + q + r)((p - q)^2 + (q - r)^2 + (r - p)^2), \end{aligned}$$

or equivalently,

$$s = \frac{t}{2}((p - q)^2 + (q - r)^2 + (r - p)^2).$$

Note that $(p-q)^2 + (q-r)^2 + (r-p)^2 \geq 0$, with equality if and only if $p = q = r$. By hypothesis, this cannot hold, so $(p-q)^2+(q-r)^2+(r-p)^2 > 0$. Thus, $t = 0$ if and only if $s = 0$. Moreover, when $s, t \neq 0$, we may divide by t to get $\frac{s}{t} = \frac{1}{2}((p - q)^2 + (q - r)^2 + (r - p)^2) > 0$, and the result follows. ■

Now, we set $p = a, q = b\sqrt[3]{2}$, and $r = c\sqrt[3]{4}$, so by our lemma,

$$p^3 + q^3 + r^3 - 3pqr = a^3 + 2b^3 + 4c^3 - 6abc$$

has the same sign as $p + q + r = a + b\sqrt[3]{2} + c\sqrt[3]{4}$, provided that $p = q = r$ does not hold. If $p = q = r$ does hold, then $a = b\sqrt[3]{2} = c\sqrt[3]{4}$, which implies $a = b = c = 0$ because a, b, c are integers. Thus, $f(a,b,c) = a^3 + 2b^3 + 4c^3 - 6abc = a + b + c = 0$, and this case is covered as well. This concludes our proof.

Solution 2 by: Laura Starkston (11/AZ)

If the signs must be the same, the zeros must be the same so...

$$\begin{aligned} a + b\sqrt[3]{2} + c\sqrt[3]{4} &= 0 \\ a\sqrt[3]{2} + b\sqrt[3]{4} + 2c &= 0 \\ -a\sqrt[3]{2} - b\sqrt[3]{4} &= 2c \end{aligned}$$

Keep this in mind. Rewrite the original equation:

$$\begin{aligned} a + b\sqrt[3]{2} + c\sqrt[3]{4} &= 0 \\ \left(a + b\sqrt[3]{2}\right)^3 &= \left(-c\sqrt[3]{4}\right)^3 \\ a^3 + 3\sqrt[3]{2}a^2b + 3\sqrt[3]{4}ab^2 + 2b^3 &= -4c^3 \\ a^3 + 2b^3 + 4c^3 &= -3\sqrt[3]{2}a^2b - 3\sqrt[3]{4}ab^2 \\ a^3 + 2b^3 + 4c^3 &= (-a\sqrt[3]{2} - b\sqrt[3]{4})(3ab) \end{aligned}$$

Combine the equations:

$$\begin{aligned} a^3 + 2b^3 + 4c^3 &= 6abc \\ a^3 + 2b^3 + 4c^3 - 6abc &= 0 \end{aligned}$$

Since the only operations performed were multiplication by a constant (which does not change the sign or the zeros, only the magnitude of the values) and cubing (which does not change the zeros; it makes each zero occur 3 times, but each zero is still the same; it preserves the sign because it is an odd number power), $f(a,b,c)$ where the function is defined as $f(x,y,z) = x^3 + 2y^3 + 4z^3 - 6xyz$ should have the same sign as $a + b\sqrt[3]{2} + c\sqrt[3]{2}$.

4/4/16. Find, with proof, all integers n such that there is a solution in nonnegative real numbers (x, y, z) to the system of equations

$$2x^2 + 3y^2 + 6z^2 = n \quad \text{and} \quad 3x + 4y + 5z = 23.$$

Credit This problem was created by USAMTS Director Richard Rusczyk.

Comments There are a variety of solutions to this problem. Zhou Fan and Nathan Pflueger present algebraic approaches. Lawrence Chan considers a geometric interpretation of the problem, and Richard McCutchen uses vectors.

Solution 1 by: Zhou Fan (11/NJ)

Let $3x = x_1$, $4y = y_1$, and $5z = z_1$. Then $x_1 + y_1 + z_1 = 23$ and $\frac{2}{9}x_1^2 + \frac{3}{16}y_1^2 + \frac{6}{25}z_1^2 = n$. We can find bounds for n as follows:

For the upper bound, we note:

$$\begin{aligned} n &= \frac{2}{9}x_1^2 + \frac{3}{16}y_1^2 + \frac{6}{25}z_1^2 \\ &\le \frac{6}{25}(x_1^2 + y_1^2 + z_1^2) \\ &\le \frac{6}{25}(x_1 + y_1 + z_1)^2 \\ &= \frac{3174}{25} \end{aligned}$$

We can obtain equality when $(x_1, y_1, z_1) = (0, 0, 23)$.

For the lower bound, we note that $\frac{207}{28} + \frac{184}{21} + \frac{575}{84} = 23$, so we can let $(x_1, y_1, z_1) = (\frac{207}{28} + a, \frac{184}{21} + b, \frac{575}{84} + c)$ where $a + b + c = 0$. Then

$$\begin{aligned} n &= \frac{2}{9}x_1^2 + \frac{3}{16}y_1^2 + \frac{6}{25}z_1^2 \\ &= \frac{2}{9}\left(\frac{207}{28} + a\right)^2 + \frac{3}{16}\left(\frac{184}{21} + b\right)^2 + \frac{6}{25}\left(\frac{575}{84} + c\right)^2 \\ &= \frac{2}{9}a^2 + \frac{23}{7}a + \frac{13}{16}b^2 + \frac{23}{7}b + \frac{6}{25}c^2 + \frac{23}{7}c + \frac{529}{14} \\ &= \frac{2}{9}a^2 + \frac{13}{16}b^2 + \frac{6}{25}c^2 + \frac{529}{14} \\ &\ge \frac{529}{14} \end{aligned}$$

We can obtain equality when $a, b, c = 0$ and thus $(x_1, y_1, z_1) = (\frac{207}{28}, \frac{184}{21}, \frac{575}{84})$.

We note that the ranges of values of x_1, y_1, and z_1 are all continuous intervals since the original x, y, z can be any nonnegative real numbers. Thus, we can have a continuous change between $(x_1, y_1, z_1) = (\frac{207}{28}, \frac{184}{21}, \frac{575}{84})$ and $(x_1, y_1, z_1) = (0, 0, 0)$, so n can take any real value between $\frac{529}{14}$ and $\frac{3174}{25}$. Thus, if n is restricted to the integers, then n can be any integer between 38 and 126 inclusive.

Solution 2 by: Nathan Pflueger (12/WA)

We shall show that all integers n, $38 \leq n \leq 126$, there exist nonnegative reals (x, y, z) such that $2x^2 + 3y^2 + 6z^2 = n$ and $3x + 4y + 5z = 23$, and these are the only integers n with this characteristic.

Define $f(x, y, z) = 2x^2 + 3y^2 + 6z^2$, for any nonnegative reals x, y, z, satisfying $3x + 4y + 5z = 23$. Note that "$f(x, y, z) = a$" carries both the implication that $2x^2 + 3y^2 + 6z^2 = a$ as well as $3x + 4y + 5z = 23$, f only being defined on its given domain.

LEMMA: If $f(x_1, y_1, z_1) = a_1$, and $f(x_2, y_2, z_2) = a_2$, then for all a between a_1 and a_2, there exist x, y, z such that $f(x, y, z) = a$.
PROOF: Define the function $g(t) = f(x_1 + (x_2 - x_1)t, y_1 + (y_2 - y_1)t, z_1 + (z_2 - z_1)t)$. The arguments given for f lie between the (x_1, y_1, z_1) and (x_2, y_2, z_2), so they are nonnegative, and they satisfy $3x + 4y + 5z = 23$, so g is defined for all $t \in [0, 1]$. g expands out to a polynomial function of t, so it is continuous. Since $g(0) = a_1$ and $g(1) = a_2$, for all a between a_1 and a_2 there is some $t \in [0, 1]$ such that $g(t) = a$, and thus some (x, y, z) such that $f(x, y, z) = a$.

By the Cauchy-Schwarz inequality, $(x\sqrt{2} \cdot \frac{3}{\sqrt{2}} + y\sqrt{3} \cdot \frac{4}{\sqrt{3}} + z\sqrt{6} \cdot \frac{5}{\sqrt{6}})^2 \leq (2x^2 + 3y^2 + 6z^2)(\frac{9}{2} + \frac{16}{3} + \frac{25}{6}) = 14 \cdot f(x, y, z)$, so $f(x, y, z) \geq \frac{1}{14}(3x + 4y + 5z)^2 = \frac{23^2}{14} = \frac{529}{14}$. By applying the equality condition of Cauchy, we find that $f(\frac{69}{28}, \frac{46}{21}, \frac{115}{84}) = \frac{529}{14}$.

By the trivial ineqality and the fact that $\frac{3}{16} < \frac{2}{9} < \frac{6}{25}$, $f(x, y, z) = \frac{2}{9}(3x)^2 + \frac{3}{16}(4y)^2 + \frac{6}{25}(5z)^2 \leq \frac{6}{15}((3x)^2 + (4y)^2 + (5z)^2)$. Since $x, y, z > 0$ and due to the expansion of $(a+b+c)^2$, this is in turn less than or equal to $\frac{6}{15}(3x + 4y + 5z)^2 = \frac{6}{15}23^2 = \frac{3174}{25}$. Thus we have $f(x, y, z) \leq \frac{3174}{25}$. The equality case is achievable; $f(0, 0, \frac{23}{5}) = \frac{3174}{25}$.

Thus we have determined that $\frac{529}{14} \leq f(x, y, z) \leq \frac{3174}{25}$, and there exist x, y, z satisfying both equality cases. Therefore, by the lemma, for any integer $n \in [\frac{529}{14}, \frac{3174}{25}]$, there exist nonnegative reals (x, y, z) such that $f(x, y, z) = n$. $\frac{529}{14} = 37\frac{11}{14}$, and $\frac{3174}{25} = 126\frac{24}{25}$, so all possible *integer* values of n are the integers from 38 to 126.

Solution 3 by: Lawrence Chan (11/IL)

There is an interesting geometric solution to this problem. We begin by noting that $2x^2 + 3y^2 + 6z^2 = n$ is an ellipsoid and $3x + 4y + 5z = 23$ is a plane. If the ellipsoid becomes too small, it will not be able to touch the plane. If it becomes too big, its intersection with the plane will be outside the set of triples of positive reals. We will begin by finding the lower bound.

Since we want to find the point where the ellipsoid just touches the plane, we want to find n such that the ellipsoid is tangent to the plane. To simplify things, we will rearrange variables and define new variables.

$$\begin{aligned} 2x^2 + 3y^2 + 6z^2 &= n \\ (\sqrt{2}x)^2 + (\sqrt{3}y)^2 + (\sqrt{6}z)^2 &= n \end{aligned}$$

We will define

$$\begin{aligned} u &:= \sqrt{2}x \\ v &:= \sqrt{3}y \\ w &:= \sqrt{6}z \end{aligned}$$

Our equation then becomes

$$u^2 + v^2 + w^2 = n$$

which is a sphere centered at (0,0,0) with radius $\sqrt{n}$.

We will also change the variables in the equation of the plane to get

$$\frac{3}{\sqrt{2}}u + \frac{4}{\sqrt{3}}v + \frac{5}{\sqrt{6}}w = 23$$

If a plane is tangent to a sphere centered at the origin, the distance from the plane to the origin is equal to the radius of the sphere. Thus,

$$\begin{aligned} \sqrt{n} &= \frac{\left|\frac{3}{\sqrt{2}}(0) + \frac{4}{\sqrt{3}}(0) + \frac{5}{\sqrt{6}}(0) - 23\right|}{\sqrt{(\frac{3}{\sqrt{2}})^2 + (\frac{4}{\sqrt{3}})^2 + (\frac{5}{\sqrt{6}})^2}} \\ n &= \frac{23^2}{14} \\ n &\approx 37.7857 \end{aligned}$$

We want the the first n that works, so our lower bound is $n = 38$. To find the upper bound, we check the axes since those are the farthest reaching (in other words, we let x and y equal 0 to find the upper bound by z, and we do this for each of the three variables). We want to find the largest n such that

$$\begin{aligned}
\sqrt{\frac{n}{6}} &= \frac{23}{5} \text{ checking z axis} \\
n &\approx 126.96 \\
\sqrt{\frac{n}{3}} &= \frac{23}{4} \text{ checking y axis} \\
n &\approx 99.19 \\
\sqrt{\frac{n}{2}} &= \frac{23}{3} \text{ checking x axis} \\
n &\approx 117.56
\end{aligned}$$

The highest n that still works is 126, so our answer is as follows:

$\boxed{\text{n}= \{x : x \in \mathbf{Z} \text{ and } 38 \le x \le 126\}}$

■

Solution 4 by: Richard McCutchen (10/MD)

A change of variables will make this problem a bit easier. Let $x = u/\sqrt{2}$, $y = v/\sqrt{3}$, and $z = w/\sqrt{6}$; the new variables u, v, and w are nonnegative iff x, y, and z are. Our two equations become

$$u^2 + v^2 + w^2 = n, \tag{1}$$

$$\frac{3}{\sqrt{2}}u + \frac{4}{\sqrt{3}}v + \frac{5}{\sqrt{6}}w = 23. \tag{2}$$

Introduce a rectangular uvw coordinate system; we consider only the region where $u, v, w \ge 0$ (henceforth the first octant). (1) asserts that the distance from (u, v, w) to the origin is $\sqrt{n}$, while (2) asserts that (u, v, w) is on a plane (call it $\mathcal{P}$) that does not depend on n. Thus, if we can determine at what distances the first-octant points on $\mathcal{P}$ lie from the origin, we'll know for which values of n the system is solvable.

For convenience, let

$$\vec{\mathbf{a}} = \left\langle \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \frac{5}{\sqrt{6}} \right\rangle \qquad \text{and} \qquad \vec{\mathbf{b}} = \langle u, v, w \rangle.$$

We can then rewrite (1) and (2) as

$$|\vec{\mathbf{b}}| = \sqrt{n}, \tag{1$'$}$$

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = 23. \tag{2$'$}$$

It is also useful to know that $|\vec{\mathbf{a}}| = \sqrt{14}$:

$$\left|\left\langle \frac{3}{\sqrt{2}}, \frac{4}{\sqrt{3}}, \frac{5}{\sqrt{6}} \right\rangle\right| = \sqrt{\left(\frac{3}{\sqrt{2}}\right)^2 + \left(\frac{4}{\sqrt{3}}\right)^2 + \left(\frac{5}{\sqrt{6}}\right)^2} = \sqrt{\frac{9}{2} + \frac{16}{3} + \frac{25}{6}} = \sqrt{14}.$$

The first-octant point of $\mathcal{P}$ nearest the origin

The Cauchy-Schwartz Inequality on vectors states that

$$(\vec{\mathbf{a}} \cdot \vec{\mathbf{b}}) \le |\vec{\mathbf{a}}| \cdot |\vec{\mathbf{b}}|, \tag{3}$$

where $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are as above. I'd like to use this to obtain a lower bound on the distance from points $\vec{\mathbf{b}}$ on $\mathcal{P}$ from the origin; this distance is $|\vec{\mathbf{b}}|$. Since $\vec{\mathbf{b}}$ is on $\mathcal{P}$, we can substitute for the left side using $(2')$, and we already know what $|\vec{\mathbf{a}}|$ is:

$$23 \le \sqrt{14}|\vec{\mathbf{b}}| \Rightarrow |\vec{\mathbf{b}}| \ge 23/\sqrt{14}. \tag{4}$$

This is interesting.

There is, in fact, a point of $\mathcal{P}$ in the first octant that lies at a distance of $23/\sqrt{14}$ from the origin. It is $\vec{\mathbf{b}} = (23/14)\vec{\mathbf{a}}$. This point lies on $\mathcal{P}$ because $(23/14)\vec{\mathbf{a}} \cdot \vec{\mathbf{a}} = (23/14)|\vec{\mathbf{a}}|^2 = 23$, and $|\vec{\mathbf{b}}| = (23/14)|\vec{\mathbf{a}}| = 23/\sqrt{14}$.

The first-octant point of $\mathcal{P}$ furthest from the origin

What first-octant point of $\mathcal{P}$ is furthest from the origin? To find this out, I would like to prove a lemma.

Lemma. Let $\mathcal{A} = A_1A_2 \ldots A_k$ be a polygon lying in 3-space, and let P be a point in 3-space. Of all the points on or inside $\mathcal{A}$, the point furthest from P is one of the vertices A_i. (If multiple points on or inside $\mathcal{A}$ tie for furthest, they are all vertices of $\mathcal{A}$.)

Proof. Let Q be any point of $\mathcal{A}$ other than a vertex; I will prove that there is a point of $\mathcal{A}$ further from P than Q. There must be a line segment $\overline{UV}$ lying on or inside $\mathcal{A}$ and having Q as its midpoint. (If Q is on a side of $\mathcal{A}$, take a short segment along this side. Otherwise, any sufficiently short segment with midpoint Q and in the plane of $\mathcal{A}$ will do.)

Let $\vec{\mathbf{w}} = \vec{QV} = \vec{UQ}$. Suppose $\vec{\mathbf{w}} \cdot \vec{PQ} \ge 0$. Then V is further from P than Q is, because

$$PV^2 = |\vec{PQ} + \vec{\mathbf{w}}|^2 = (\vec{PQ} + \vec{\mathbf{w}}) \cdot (\vec{PQ} + \vec{\mathbf{w}}) = PQ^2 + |\vec{\mathbf{w}}|^2 + 2PQ|\vec{\mathbf{w}}| > PQ^2 + 2PQ|\vec{\mathbf{w}}| \ge PQ^2.$$

(The last inequality follows because $\vec{\mathbf{w}} \cdot \vec{PQ} \ge 0$.) If $\vec{\mathbf{w}} \cdot \vec{PQ} < 0$, the proof is similar; U is further from P than Q is. Either way, we have proved the lemma. □

It should be clear that the intersection of the fixed plane $\mathcal{P}$ with the first octant is a triangle (call it T) with one vertex on each axis. Using (2), these vertices are

$$\left(\frac{23\sqrt{2}}{3}, 0, 0\right), \quad \left(0, \frac{23\sqrt{3}}{4}, 0\right), \quad \left(0, 0, \frac{23\sqrt{6}}{5}\right).$$

A quick calculator approximation shows that $(0, 0, 23\sqrt{6}/5)$ is the furthest from the origin among these three. The lemma then says that $(0, 0, 23\sqrt{6}/5)$ is furthest from the origin among all points of T. Of course, the distance is $23\sqrt{6}/5$.

Putting it together

So, we know that, of the points of the triangle T, $(23/14)\vec{\mathbf{a}}$ ($\vec{\mathbf{a}}$ was the messy vector from a while ago) is the closest to the origin at a distance of $23\sqrt{14}$, and $(0, 0, 23\sqrt{6}/5)$ is the furthest from the origin at a distance of $23\sqrt{6}/5$. Now, back to the system

$$u^2 + v^2 + w^2 = n, \tag{1}$$

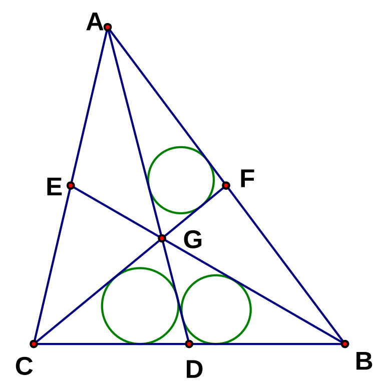
$$\frac{3}{\sqrt{2}}u + \frac{4}{\sqrt{3}}v + \frac{5}{\sqrt{6}}w = 23. \tag{2}$$

(2) says that (u, v, w) is on or inside T. (1) says that (u, v, w) is at a distance $\sqrt{n}$ from the origin. Thus, the system is solvable for a given n iff T has a point $\sqrt{n}$ units from the origin. Thus, if n could be a real number, it could range from

$$(23/\sqrt{14})^2 = 529/14 \approx 37.78 \quad \text{to} \quad (23\sqrt{6}/5)^2 = 3174/25 = 126.96.$$

Since n must be an integer, the system is solvable for $38 \leq n \leq 126$. ■

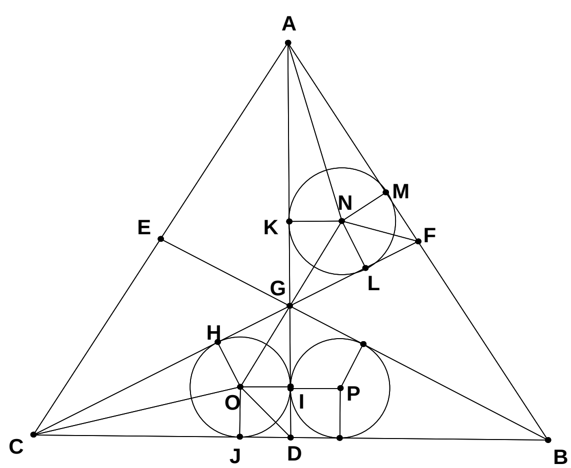
5/4/16. Medians AD, BE, and CF of triangle ABC meet at G as shown. Six small triangles, each with a vertex at G, are formed. We draw the circles inscribed in triangles AFG, BDG, and CDG as shown. Prove that if these three circles are all congruent, then ABC is equilateral.



Credit This problem was contributed by Professor Gregory Galperin, a long-time contributor of problems to the USAMTS.

Comments Most solutions involved first showing that $\triangle CGD \cong \triangle BGD$ by first showing that the perimeters of these triangles are equal. Students took a variety of approaches to showing $AF = CD$, some using a purely geometric approach, as Benjamin Dozier illustrates, some using a more trigonometric approach like that of Shotaro Makisumi, and some using a little mix of the two, like Dan Li does. *Solutions edited by Richard Rusczyk*

Solution 1 by: Benjamin Dozier (9/NM)



The area of $\triangle CDG$ equals the area of $\triangle BDG$ as they share the altitude from G to $\overline{BC}$ and they have bases of equal length. $\triangle CDG$ can be dissected into $\triangle OCD$, $\triangle ODG$ and $\triangle COG$. Likewise, $\triangle BDG$ can be dissected into $\triangle PBD$, $\triangle PDG$ and $\triangle BPG$. The area of $\triangle OCD$ equals the area of $\triangle PBD$ because they share a base and the respective altitudes to that base are of the same length. Likewise, the area of $\triangle ODG$ equals the area of $\triangle PDG$. Thus the area of $\triangle COG$ equals the area of $\triangle BPG$. Since these two triangles have altitudes of the same length, they must have bases of the same length. Therefore $CG = BG$. We know $CD = DB$, so $\triangle CDG \cong \triangle BDG$ by Side-Side-Side congruence. Furthermore, $m\angle GDC + m\angle GDB = 180°$ and so $m\angle GDC = m\angle GDB = 90°$. Since median $\overline{AD}$ is also the altitude, we know that $\triangle ABC$ is isosceles with $AC = AB$.

Now, since O and N lie on the angle bisectors of $\angle CGD$ and $\angle AGF$ respectively, and $\angle CGD = \angle AGD$, we know that $\angle OGI = \angle NGL$. Also, $NL = OI$ and both $\angle OIG$ and $\angle GLN$ are right, so $\triangle OGI \cong \triangle NGL \cong \triangle OGH \cong \triangle NGK$. Now $\triangle OCH \cong \triangle OCJ$ by ASA congruence. Likewise $\triangle ODI \cong \triangle ODJ$, $\triangle NLF \cong \triangle NMF$, and $\triangle NAM \cong \triangle NAK$. All of these triangles have an altitude of common length, the inradius, which we will call r. The area of $\triangle CDG$ is the same as the area of $\triangle GFA$ as the medians dissect a triangle into six smaller triangles all of the same area. Thus:

$$(2)(\frac{1}{2}r)GH + (2)(\frac{1}{2}r)JD + (2)(\frac{1}{2}r)CJ = (2)(\frac{1}{2}r)GK + (2)(\frac{1}{2}r)AM + (2)(\frac{1}{2}r)MF$$

Since $GH = GK$:

$$\begin{aligned} (r)JD + (r)CJ &= (r)AM + r(MF) \\ JD + CJ &= AM + MF \\ AF &= CD \end{aligned}$$

which implies that $AB = AC = CB$ and thus the triangle is equilateral.

Solution 2 by: Shotaro Makisumi (9/CA)

Since the centroid divides each median into segments of proportion $1:2$, each of the six small triangles has a base that is half of and a height a third of $\triangle ABC$, and so they all have the same area. We know that the radii of the incircles of $\triangle AFG$, $\triangle BDG$, and $\triangle CDG$ are equal. Since $2A = rp$, where A is the area of the triangle, p is the perimeter, and r is the radius of the incircle, are all equal, these triangles all have equal perimeter. That is,

$$CD + CG + DG = BD + BG + DG = AF + AG + FG \tag{1}$$

But $CD = BD$, so $CG = BG$. By SSS congruence, $\triangle CDG \cong \triangle BDG$. This implies $\angle CDG = 90°$.

We let $x = FG$ and $\theta = m\angle CGD = m\angle AGF$. Then we have $CG = 2x$. Since $\triangle CDG$ is a right triangle, $CD = 2x\sin\theta$ and $DG = 2x\cos\theta$, and so $AG = 2DG = 4x\cos\theta$. We apply the Law of Cosines on $\triangle AFG$:

$$\begin{aligned}(AF)^2 &= (FG)^2 + (AG)^2 - 2(FG)(AG)\cos(m\angle AGF)\\ (AF)^2 &= x^2 + (4x\cos\theta)^2 - 2x(4x\cos\theta)\cos\theta\\ (AF)^2 &= x^2 + 8x^2\cos^2\theta\\ AF &= x\sqrt{1+8\cos^2\theta}\end{aligned}$$

Now we can rewrite the second equality of (1) as follows:

$$2x + 2x\cos\theta + 2x\sin\theta = x\sqrt{1+8\cos^2\theta} + 4x\cos\theta + x$$

Since $x \neq 0$, we can divide through by x and simplify:

$$\begin{aligned}1 + 2\sin\theta - 2\cos\theta &= \sqrt{1+8\cos^2\theta}\\ 4\sin^2\theta + 4\sin\theta + 1 + 4\cos^2\theta - 4\cos\theta - 8\sin\theta\cos\theta &= 1 + 8\cos^2\theta\\ -2\cos^2\theta - \cos\theta + 1 + \sin\theta - 2\sin\theta\cos\theta &= 0\\ (\sin\theta + \cos\theta + 1)(1 - 2\cos\theta) &= 0\end{aligned}$$

This is satisfied when $\sin\theta + \cos\theta + 1 = 0$ or $1 - 2\cos\theta = 0$. For $\theta \in (0^\circ, 90^\circ)$, the former has no solution, since $\sin\theta > 0$ and $\cos\theta > 0$. We solve the second equation to obtain

$$\cos\theta = \frac{1}{2}$$

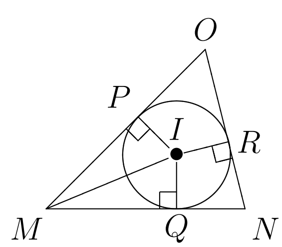
Finally, $AG = 4x\cos\theta = 2x = CG = BG$. Since the longer portions of the medians are congruent, the shorter portions are also congruent, and all six smaller triangles are congruent. This occurs only if $\triangle ABC$ is equilateral.

Q.E.D.

Solution 3 by: Dan Li (10/CA)

Lemma 5.1. *The inradius, r, of a triangle with sides m, n, o and angle μ opposite the side of length m is* $r = \left(\frac{n+o-m}{2}\right)\left(\tan\frac{\mu}{2}\right)$.

Proof. Let the triangle be $\triangle MNO$, with $NO = m$, $MN = n$, $MO = o$, and $\angle NMO = \mu$. Let the incenter of $\triangle MNO$ be I. Let the points of tangency on $\overline{MO}$, $\overline{MN}$, $\overline{NO}$ be P, Q, R, respectively.



Because I is equidistant from $\overline{MO}$ and $\overline{MN}$ ($IP = IQ = r$), it lies on the bisector of $\angle NMO$. Therefore,

$$\angle IMQ = \frac{\angle NMO}{2} = \frac{\mu}{2} \tag{2}$$

Because the segments from one point to two points of tangency have equal length (e.g. $MP = MQ$),

$$\begin{aligned} MQ &= n - QN = n - RN = n - (m - RO) \\ &= n - m + PO = n - m + (o - MP) = n - m + o - MQ \quad (3) \\ 2MQ &= n + o - m \quad (4) \\ MQ &= \frac{n + o - m}{2} \quad (5) \end{aligned}$$

Thus,

$$r = IQ = MQ(\tan \angle IMQ) = \left(\frac{n + o - m}{2}\right)\left(\tan\frac{\mu}{2}\right) \tag{6}$$

□

Let $i = GD$, $j = GE$, $k = GF$, $a = FA$, $b = EA$, $c = DB$. Because the distance from the centroid (G) to a vertex is twice the distance from the centroid to the midpoint of the side opposite the vertex, $GA = 2i$, $GB = 2j$, $GC = 2k$. By the definition of median, $FB = a$, $EC = b$, $DC = c$.

Let $\alpha = \angle CGD = \angle AGF$. Because the inradii of $\triangle CGD$ and $\triangle AGF$ are equal and by Lemma 5.1,

$$\begin{aligned} \left(\frac{CG + GD - DC}{2}\right)\left(\tan\frac{\angle CGD}{2}\right) &= \left(\frac{AG + GF - FA}{2}\right)\left(\tan\frac{\angle AGF}{2}\right) \quad (7) \\ \left(\frac{2k + i - c}{2}\right)\left(\tan\frac{\alpha}{2}\right) &= \left(\frac{2i + k - a}{2}\right)\left(\tan\frac{\alpha}{2}\right) \quad (8) \\ 2k + i - c &= 2i + k - a \quad (9) \\ k + a &= i + c \quad (10) \end{aligned}$$

It is well-known that "all the medians together divide [a triangle] into six equal parts" (http://mathworld.wolfram.com/TriangleCentroid.html). Therefore, the areas of $\triangle AGF$, $\triangle CGD$, and $\triangle BGD$ are equal. It is similarly well-known that the product of the semi-perimeter and the inradius equals the area of a triangle (see (7) at `http://mathworld.wolfram.com/TriangleArea.html`; a proof is given at `http://mathworld.wolfram.com/Inradius.html`). Since the inradii of the three triangles are equal (since their incircles are congruent) and the areas are equal, their semiperimeters must be equal. Therefore,

$$\frac{2i+k+a}{2} = \frac{2k+i+c}{2} = \frac{2j+i+c}{2} \tag{11}$$

The second and third parts of (11) result in

$$\frac{2k+i+c}{2} = \frac{2j+i+c}{2} \tag{12}$$

$$k = j \tag{13}$$

Therefore, $\triangle EGC \cong \triangle FGB$ by SAS ($EG = j = k = FG$, $CG = 2k = 2j = BG$, $\angle EGC \cong \angle FGB$). Hence, $EC = FB$ and

$$b = a \tag{14}$$

The first and second parts of (11) yield

$$\frac{2i+k+a}{2} = \frac{2k+i+c}{2} \tag{15}$$

$$k + c = i + a \tag{16}$$

Subtracting (10) from (16) yields

$$c - a = a - c \tag{17}$$

$$c = a \tag{18}$$

Combining (14) and (18),

$$\begin{aligned} a &= b = c \\ 2a &= 2b = 2c \\ AB &= AC = BC \end{aligned}$$

Hence, $\triangle ABC$ is equilateral.