1/3/16. Given two integers x and y, let $(x\|y)$ denote the *concatenation* of x by y, which is obtained by appending the digits of y onto the end of x. For example, if $x = 218$ and $y = 392$, then $(x\|y) = 218392$.
(a) Find 3-digit integers x and y such that $6(x\|y) = (y\|x)$.
(b) Find 9-digit integers x and y such that $6(x\|y) = (y\|x)$.

Credit The 3-digit variety of the problem was inspired by Problem 28 in the Singapore Mathematical Olympiad (Junior Section) in 2001. The 9-digit extension is due to USAMTS founder Dr. George Berzsenyi.

Comments Many students took a trial-and-error appraoch. The most common algebraic approach to part (a) is reflected in Jason Bland's solution. Many students used this approach for part (b), but a few students used the slick approach of using (a) to get (b) as shown in Nathan Pflueger's solution below. Still others used the number 1,000,001,000,001 as Jason Bland illustrates below. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Nathan Pflueger (12/WA)

(a)
Let $(x, y) = (142, 857)$. Multiplication yields $6(x||y) = 6 \cdot 142857 = 857142 = (y||x)$.

(b)
Let $(x, y) = (142, 857)$ as above. Let $(u, v) = (x||y||x, y||x||y)$. It was shown above that $6(x||y) = (y||x)$ thus $6(u||v) = 6(x||y||x||y||x||y) = (y||x||y||x||y||x) = (v||u)$, thus u and v are the 9-digit integers we seek: 142857142 and 857142857, respectively. Alternating concatenations such as this can also be used to select two such integers for any number of digits of the form $3 + 6n$.

Solution 2 by: Jason Bland (10/PA)

(a) Because x and y each have 3 digits, we can write $(x||y) = 1000x + y$. Therefore, we have

$$\begin{aligned} 6(1000x+y) &= 1000y + x \\ 6000x + 6y &= 1000y + x \\ 5999x &= 994y \\ 857x &= 142y \\ x = 142 & \quad y = 857 \end{aligned}$$

(b) $(x||y)$ has 6 digits when x and y have 3 digits each and 18 digits when x and y have 9 digits each, so multiplying the equation involving $(x||y)$ and $(y||x)$ for 3-digit x and y by 1,000,001,000,001 gives the equation involving $(x||y)$ and $(y||x)$ for 9-digit x and y.

$$\begin{aligned} 6 * 142{,}857 &= 857{,}142 \\ 6 * 142{,}857{,}142{,}857{,}142{,}857 &= 857{,}142{,}857{,}142{,}857{,}142 \\ x = 142{,}857{,}142 & \quad y = 857{,}142{,}857 \end{aligned}$$

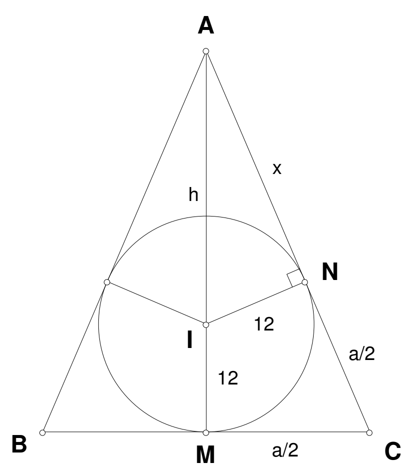
2/3/16. Find three isosceles triangles, no two of which are congruent, with integer sides, such that each triangle's area is numerically equal to 6 times its perimeter.

Credit This is a slight modification of a problem provided by Suresh T. Thakar of India. The original problem asked for five isosceles triangles with integer sides such that the area is numerically 12 times the perimeter.

Comments Many students simply set up an equation using Heron's formula and then turned to a calculator or a computer for a solution. Below are presented more elegant solutions. Zachary Abel shows how to reduce this problem to finding Pythagorean triples which have 12 among the side lengths. Adam Hesterberg gives us a solution using Heron's formula. Finally, Kristin Cordwell shows how to take an intelligent trial-and-error approach to construct the solutions. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Zachary Abel (11/TX)

In $\triangle ABC$ with $AB = AC$, we use the common notations $r =$ inradius, $s =$ semiperimeter, $p =$ perimeter, $K =$ area, $a = BC$, and $b = AC$. The diagram shows triangle ABC with its incircle centered at I and tangent to BC and AC at M and N respectively.



The area of the triangle is given by $rs = K = 6p = 12s$, which implies $r = 12$. The area is also

$$K = \frac{a}{2} \cdot AM = \frac{a}{2} \cdot \sqrt{AC^2 - MC^2} = \frac{a}{4}\sqrt{4b^2 - a^2}.$$

Since K is an integer (since it is six times the perimeter), $4b^2 - a^2$ must be a perfect square. If a were odd, then $4b^2 - a^2 \equiv 3 \bmod 4$, which is not possible for a perfect square. Thus, a is even. So $x = AN = b - \frac{a}{2}$ is an integer, and so is $h = AM - 12 = \frac{1}{2}\sqrt{4b^2 - a^2} - 12$. So, since ANI is a right triangle, the integers x, 12, and h form a Pythagorean triple.

It is easy to check that 12 can be the leg in only four Pythagorean triangles: $(5, 12, 13)$, $(12, 16, 20)$, $(9, 12, 15)$, and $(12, 35, 37)$. So these give all the possibilities for x and h.

Similar triangles ANI and AMC show that $\frac{x}{12} = \frac{h+12}{(a/2)}$, i.e. $a = \frac{24(h+12)}{x}$, which gives $a = 120, 48, 72, \frac{168}{5}$ for the four respective cases. Since a is an integer, the fourth case doesn't work. Now, since $b = x + \frac{a}{2}$, we find that $b = 65, 40, 45$ respectively for the three remaining cases. Therefore, the three triangles have dimensions $(120, 65, 65)$, $(48, 40, 40)$, and $(72, 45, 45)$, and there are no others that satisfy the conditions of the problem.

Solution 2 by: Adam Hesterberg (10/WA)

Answers: Triangles with sides (72,45,45), (48,40,40), (120,65,65).

Let the sides of the triangle be (a, b, b). Then the six times the perimeter of the triangle is $6a + 12b$, and the area of the triangle, by Heron's formula, is $\sqrt{\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(b-\frac{a}{2}\right)}$.

$$\begin{aligned}
6a + 12b &= \sqrt{\left(b+\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(\frac{a}{2}\right)\left(b-\frac{a}{2}\right)} \\
144\left(b+\frac{a}{2}\right)^2 &= \left(\frac{a}{2}\right)^2\left(b-\frac{a}{2}\right)\left(b+\frac{a}{2}\right) \\
144b + 72a &= \left(\frac{a}{2}\right)^2\left(b-\frac{a}{2}\right) \\
\frac{2b+a}{2b-a} &= \left(\frac{a}{24}\right)^2
\end{aligned}$$

Trying multiples of 24 for a leads to ($a = 48$, $b = 40$), ($a = 72$, $b = 45$), and ($a = 120$, $b = 65$). These have areas of 972, 768, and 1500, respectively, all of which are 6 times their perimeters.

Solution 3 by: Kristin Cordwell (8/NM)

Three isosceles triangles whose area equals six times their perimeter are 45 by 45 by 72, 40 by 40 by 48, and 65 by 65 by 120.

To begin with, we notice that two congruent right triangles stuck together at a common leg form an isosceles triangle. We then consider common right triangles: 3,4,5 and 5,12,13. The other feature that we need to note is that, if we scale the perimeter by a factor α, then the area scales by α^2.

If we take two 3,4,5 right triangles and join them along the short side, we get an isosceles triangle of $P = 18$, and $A = 12$. Since $P = 2 \cdot 3^2$ and $A = 2^2 \cdot 3$, we see that, if we scale P by 3, A will scale by 3^2, and they will be in the proportion $A :: P = 2^2 \cdot 3^3 :: 2 \cdot 3^3$. This isn't quite what we want, but if we scale P by 3 one more time, we will end up with $2^2 \cdot 3^5 :: 2 \cdot 3^4$, or $A = 6P$.

If we take two 3,4,5 right triangles and join them along the long side, we get an isosceles triangle of $P = 16$, and $A = 12$. Note that we already have a factor of 3 in the A to P ratio, but that we need another net factor of 2^3 to get the overall ratio of 6. Since we "gain" a net factor of 2 for every doubling of the perimeter, we scale P by 8, which gives $P = 128$ and $A = 768 = 6P$.

Finally, if we consider two 5,12,13 triangles and glue them together at the short leg, we have a triangle of sides 13, 13, and 24, with $P = 50 = 2 \cdot 5^2$, and $A = 60 = 2^2 \cdot 3 \cdot 5$. The powers of 2 and 3 are what we wish, but we need to scale P by 5 (and A by 5^2, in order to have the powers of 5 balance. We then obtain $P = 250$ and $A = 1500 = 6P$.

In some cases, we can scale by a fraction. For example, if we look at two 7,24,25 right triangles joined at the short edge, we have $P = 98 = 2 \cdot 7^2$ and $A = 168 = 2^3 \cdot 3 \cdot 7$. If we scale P by $\frac{7}{2}$, we get a new perimeter of $P = 7^3$ and an area $A = 2 \cdot 3 \cdot 7^3$. This works because we have a starting power of 2 in the perimeter that can be canceled.

3/3/16. Define the recursive sequence 1, 4, 13, ... by $s_1 = 1$ and $s_{n+1} = 3s_n + 1$ for all positive integers n. The element $s_{18} = 193710244$ ends in two identical digits. Prove that all the elements in the sequence that end in two or more identical digits come in groups of three consecutive elements that have the same number of identical digits at the end.

Credit This problem was devised by Erin Schram of the NSA. It is based on an "Olympiad Problem" of a 2003 issue of the *Gazeta Matematica* magazine that was posted on the Art of Problem Solving forum.

Comments Many students proved that the last two digits repeat in a cycle of 20, and used this cycle to prove that the elements in the sequence that end in two or more identical digits come in groups of three consecutive elements. Fewer students proved the second half – that within each of these groups of three, the three numbers have the identical number of repeating digits at the end. Jeffrey Manning gives a clear, concise explanation, and Cary Malkiewich gives us a more formal solution. *Solutions edited by Richard Rusczyk.*

Solution 1 by: Jeffrey Manning (9/CA)

If a number ends in two or more identical digits its last two digits must be identical. Working out the sequence modulo 100 gives:

$$1, 4, 13, 40, 21, 64, 93, 80, 41, 24, 73, 20, 61, 84, 53, 60, 81, \mathbf{44}, \mathbf{33}, \mathbf{00}, 1, \ldots$$

Since the sequence is recursive and $s_{21} \equiv s_1 \equiv 1 \bmod 100$ the sequence modulo 100 must repeat every 20 elements which means that all elements that end in two or more identical digits come in groups of three consecutive elements, where the digits are 4s in the first element, 3s in the second and 0s in the third. Now we must prove that they end in the same number of identical digits.

Let n be the number of 4s a the end of some element of the sequence. Since,

$$3(\underbrace{444\ldots4}_{n\ digits}) + 1 = 1\underbrace{333\ldots3}_{n\ digits} \quad \text{and} \quad 3(\underbrace{333\ldots3}_{n\ digits}) + 1 = 1\underbrace{000\ldots0}_{n\ digits}$$

each element must end in at least as many identical digits as the previous element.

For the second element to end in more than n identical digits the last $n+1$ digits of the first element must be $X\underbrace{444\ldots4}_{n\ digits}$, where X is a digit other than 4 such that $3X + 1 \equiv 3 \bmod 10$, but the only single digit that would satisfy this is 4 which is a contradiction. This means that the second element must end in exactly n identical digits.

Similarly, for the third element to end in more than n digits the last $n+1$ digits of the second element must be $Y\underbrace{333\ldots3}_{n\ digits}$ where $Y \neq 3$ and $3Y+1 \equiv 0 \mod 10$, but similarly the only single digit that would satisfy this is 3 which is a contradiction. So all three elements must end in exactly n elements. The proof is complete.

Solution 2 by: Cary Malkiewich (12/MA)

Define $\varphi_0 : \mathbf{Z}_{10} \to \mathbf{Z}_{10}$ and $\varphi_1 : \mathbf{Z}_{10} \to \mathbf{Z}_{10}$ as follows:

$$\begin{aligned}\varphi_0(a) &= 3a \mod 10\\ \varphi_1(a) &= 3a+1 \mod 10\end{aligned}$$

Lemma: The functions φ_0 and φ_1 are bijective.
Proof: This is proven simply by listing out elements.

$\varphi_0(0)=0$	$\varphi_0(5)=5$	$\varphi_1(0)=1$	$\varphi_1(5)=6$
$\varphi_0(1)=3$	$\varphi_0(6)=8$	$\varphi_1(1)=4$	$\varphi_1(6)=9$
$\varphi_0(2)=6$	$\varphi_0(7)=1$	$\varphi_1(2)=7$	$\varphi_1(7)=2$
$\varphi_0(3)=9$	$\varphi_0(8)=4$	$\varphi_1(3)=0$	$\varphi_1(8)=5$
$\varphi_0(4)=2$	$\varphi_0(9)=7$	$\varphi_1(4)=3$	$\varphi_1(9)=8$

Since every element of $\mathbf{Z}_{10}$ appears exactly once in the range of each function, each function is bijective. ■

As a result of this lemma, we can define φ_0^{-1} and φ_1^{-1} to be the inverses of the above functions.

Since $\varphi_1(1)=4$, $\varphi_1(4)=3$, $\varphi_1(3)=0$, and $\varphi_1(0)=1$, the units digit in the given sequence cycles through 1,4,3,0. These are only 4 numbers that could form the repeating digits at the end of s_n.

In order to rigorously prove the assertion, we must prove all three of these statements ($k>1$):
1. Iff s_n ends in exactly k 4's, s_{n+1} ends in exactly k 3's.
2. Iff s_{n+1} ends in exactly k 3's, s_{n+2} ends in exactly k 0's.
3. s_n can never end in two or more 1's.

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1. Iff s_n ends in exactly k 4's, s_{n+1} ends in exactly k 3's.

Assume a term s_n of the sequence ends in a string of 4's that is exactly k digits long: $(k > 1)$

$$\dots x444\dots 444$$

When s_n is multiplied by 3, every 4 becomes a 12. A 1 is carried in every column, resulting in a string of $(k-1)$ 3's followed by a 2:

$$\dots[\varphi_1(x)]333\dots 332$$

Then 1 is added, and the new number $3s_n + 1 = s_{n+1}$ ends in a string of 3's that is k digits long:

$$\dots[\varphi_1(x)]333\dots 333$$

The $(k+1)$ digit from the right, $\varphi_1(x)$, is not a 3. Since φ_1 is bijective, this would imply that the digit x in s_n is $\varphi_1^{-1}(3) = 4$, and we have assumed that only the last k digits were 4.

For the converse, suppose that s_{n+1} ends in exactly k 3's. s_n ends in a 4, so suppose s_n ends with exactly m 4's. By the argument above, s_{n+1} ends in exactly m 3's. Therefore $m = k$, and s_n ends with exactly k 4's. ■

2. Iff s_{n+1} ends in exactly k 3's, s_{n+2} ends in exactly k 0's.

Suppose s_{n+1} ends in a string of 3's that is exactly k digits long: $(k > 1)$

$$\dots y333\dots 333$$

Then $3s_{n+1}$ will end in a string of 9's that is exactly k digits long:

$$\dots[\varphi_0(y)]999\dots 999$$

$3s_{n+1} + 1 = s_{n+2}$ will then end in a string of 0's that is k digits long:

$$\dots[\varphi_1(y)]000\dots 000$$

The $(k+1)$ digit from the right, $\varphi_1(y)$, is not a 0. This would imply that the digit y in s_{n+1} is $\varphi_1^{-1}(0) = 3$, and we have assumed that only the last k digits were 3.

For the converse, suppose that s_{n+2} ends in exactly k 0's. s_{n+1} ends in a 3, so suppose s_{n+1} ends with exactly m 3's. By the argument above, s_{n+2} ends in exactly m 0's. Therefore $m = k$, and s_{n+1} ends with exactly k 3's. ■

3. s_n can never end in two or more 1's.

The last two digits form this repeating sequence of 20 terms:

$$01, 04, 13, 40, 21, 64, 93, 80, 41, 24, 73, 20, 61, 84, 53, 60, 81, 44, 33, 00$$

Since 11 is not a member of this sequence, no member of the original sequence s_n can end in two or more 1's. ■

Now that all three statements have been proven, it follows that all elements in the sequence that end in two or more identical digits (4, 3, 0) come in groups of three consecutive elements ($44\dots44, 33\dots33, 00\dots00$) that have the same number (k) of identical digits at the end.

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Solutions to Problem 4/3/16

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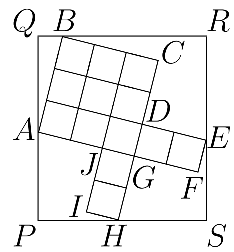
4/3/16. Region $ABCDEFGHIJ$ consists of 13 equal squares and is inscribed in rectangle $PQRS$ with A on $\overline{PQ}$, B on $\overline{QR}$, E on $\overline{RS}$, and H on $\overline{SP}$, as shown in the figure on the right. Given that $PQ = 28$ and $QR = 26$, determine, with proof, the area of region $ABCDEFGHIJ$.

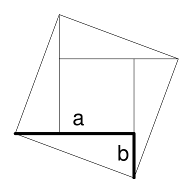


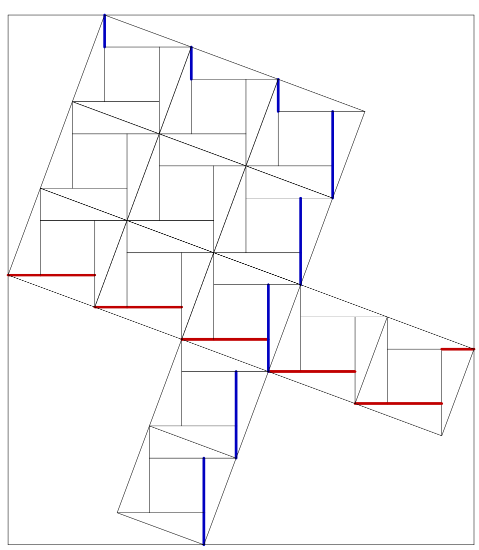
Credit This problem was inspired by Problem 2 of the First Round of the 2001 Japanese Mathematical Olympiad.

Comments There were many different successful approaches to solving this problem. The most aesthetically pleasing comes from Zachary Abel (11/TX). Others took a similar approach, using projection without Zachary's clever dissection. Noah Cohen gives an example of this approach. Nicholas Zehender shows that a subset of the figure formed by the little squares could itself be inscribed in a square and used this fact to solve the problem. Finally, Michael John Griffin gives us another dissection to solve the problem.

Solution 1 by: Zachary Abel (11/TX)

Inside each square in the diagram, draw two horizontal segments and two vertical segments as shown to the right. Let the two indicated lengths be a and b. The whole diagram looks like this:





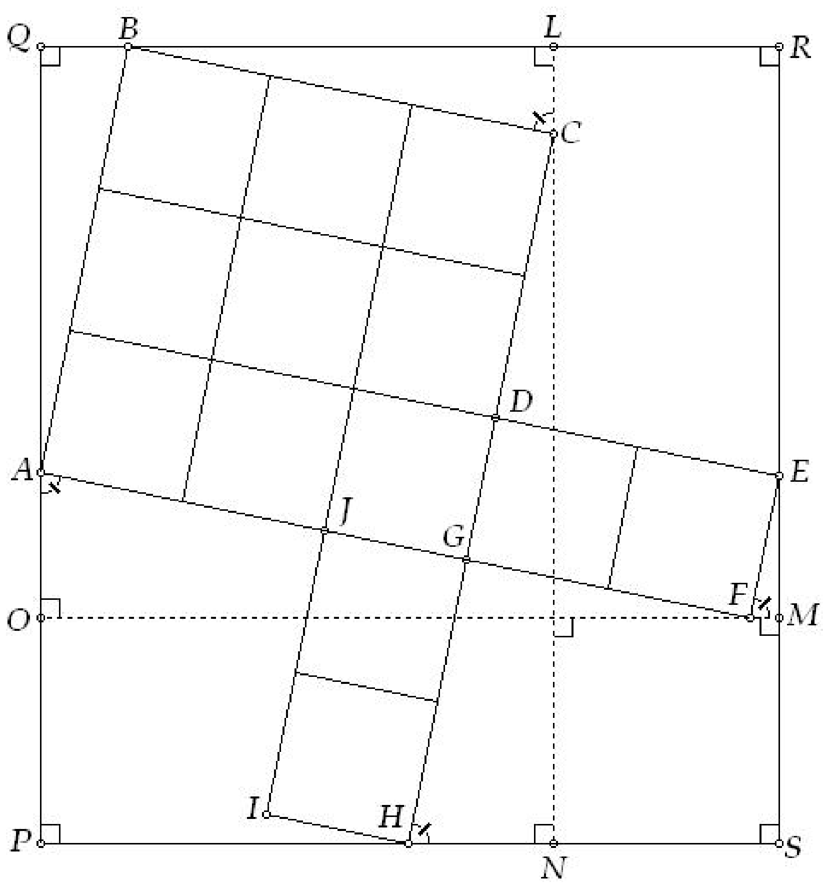
The total horizontal length of the red segments is $5a + b$, which is equal to the width of the rectangle, i.e. $5a + b = 26$. Likewise, the total vertical length of the blue segments is

$5a + 3b$, which is equal to the height of the rectangle, 28. So we have the system

$$\begin{cases} 5a + b &= 26 \\ 5a + 3b &= 28 \end{cases},$$

whose solution is $a = 5$ and $b = 1$. So the side length of each square is $\sqrt{a^2 + b^2} = \sqrt{26}$, the area of each square is 26, and the total area of $ABCDEFGHIJ$ is $26 \times 13 = 338$.

Solution 2 by: Noah Cohen (11/ME)



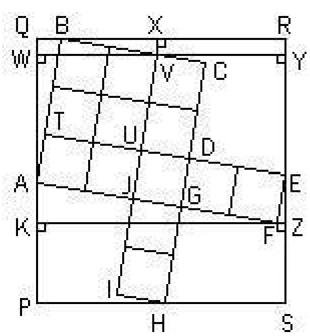
Via angle chasing, it can be seen that $\angle BCL \equiv \angle GHN \equiv \angle OAJ \equiv \angle EFM$, call this angle ϑ, and call the length of one square x

We can now represent the height and and width of the rectangle with the following equations

$$\begin{aligned} 5x\sin\vartheta + x\cos\vartheta &= 26 \\ 5x\sin\vartheta + 3x\cos\vartheta &= 28 \\ x\cos\vartheta &= 1 \\ x\sin\vartheta &= 5 \\ (x\sin\vartheta)^2 + (x\cos\vartheta)^2 &= x^2 \\ x &= \sqrt{26} \end{aligned}$$

The area of the polygon $ABCDEFGHIJ$ can be represented by $13x^2$, or $(13)(\sqrt{26})^2$, which is equal to 338.

Solution 3 by: Nicholas Zehender (11/VA)



Draw a line parallel to QR through point V. $CDEFGHIJATUV$ is the same if you rotate it 90 degrees, so $YS = WY = QR = 26$. $RY = RS - YS = 28 - 26 = 2$, so $XV = 2$. $AFK \sim VBX$ because $AF \parallel VB$, $FK \parallel BX$, and $KA \parallel XV$. $\angle EFZ = 180 - 90 - \angle AFK = 90 - \angle AFK = \angle FAK$, and $\angle EZF = \angle FKA = 90$, so FEZ is also similar to AFK and VBX.

$ZF/XV = EF/BV$
$ZF/2 = 1/2$
$ZF = 1$

$FK = ZK - ZF = 26 - 1 = 25$

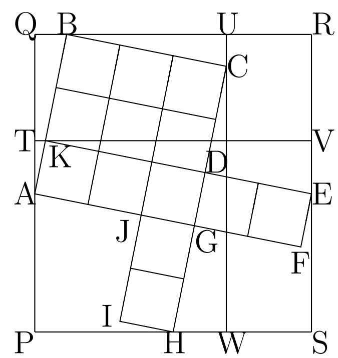
$KA/XV = AF/BV$
$KA/2 = 5/2$
$KA = 5$

$AF = \sqrt{FK^2 + KA^2}$
$AF = \sqrt{625 + 25}$
$AF = \sqrt{650}$

The side of one of the squares is $AF/5 = \sqrt{26}$, so the area of $ABCDEFGHIJ$ is $13(\sqrt{26})^2 = 338$.

Solution 4 by: Michael John Griffin (12/UT)

Let Point K be the point one third of the way between points A and B so that it is also in line with point E. Also, let points T, U, V, and W be points along lines $\overline{PQ}$, $\overline{QR}$, $\overline{RS}$, and $\overline{SP}$ respectively, such that line $\overline{TV}$ is perpendicular to line $\overline{PQ}$ and passes through point K, and line $\overline{UW}$ is perpendicular to line $\overline{QR}$ and passes through point C, as shown in the figure at right.



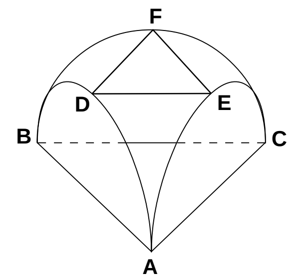
Triangles ΔABQ, ΔBCU, ΔAKT, ΔKEV, and ΔCHW are all similar, with $\Delta ABQ \cong \Delta BCU$ and $\Delta KEV \cong \Delta CHW$. All of these triangles have one right angle and the other corresponding angles equal. If two given angles (such as $\angle ABQ$ and $\angle CBU$) and a right angle are all collinear, the two angles are complimentary. In the case of ΔAKT and ΔABQ, Euclid's Coresponding Angles postulate works well.

$\overline{KV} = \overline{CW}$, $\overline{UC} = \overline{QB}$, and $\overline{TK} = 1/3\overline{QB}$ (given the ratio of the hypotenuses similar triangles). Notice that $\overline{UC} + \overline{CW} = 28$ and $\overline{TK} + \overline{KV} = 26$.

$$\begin{aligned}
\overline{UC} + \overline{CW} &= \overline{TK} + \overline{KV} + 2 \\
\overline{QB} + \overline{CW} &= \frac{\overline{QB}}{3} + \overline{CW} + 2 \\
\frac{2 * \overline{QB}}{3} &= 2 \\
\overline{QB} &= 3
\end{aligned}$$

Since $\overline{UC} = \overline{QB}$ and $\overline{UC} + \overline{CW} = 28$, $\overline{CW} = 25$. $\overline{CW} = \frac{5*\overline{AQ}}{3}$, so $\overline{AQ} = 15$. $\overline{AQ}^2 + \overline{QB}^2 = \overline{AB}^2$, so $\overline{AB}^2 = 15^2 + 3^2 = 225 + 9 = 234$. $\overline{AB}^2$ just happens to be the area of 9 of the 13 squares, so the total area of all the squares is $13/9 * 234 = 338\ units^2$

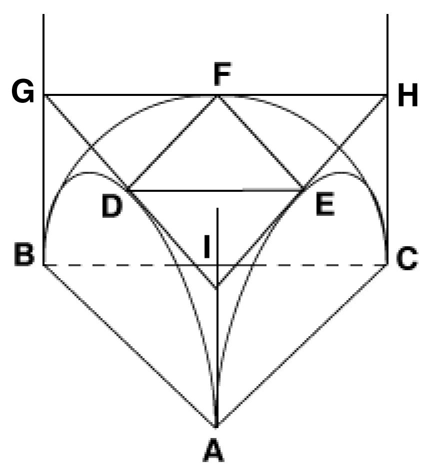
5/3/16. Consider an isosceles triangle ABC with side lengths $AB = AC = 10\sqrt{2}$ and $BC = 10\sqrt{3}$. Construct semicircles P, Q, and R with diameters AB, AC, BC respectively, such that the plane of each semicircle is perpendicular to the plane of ABC, and all semicircles are on the same side of plane ABC as shown. There exists a plane above triangle ABC that is tangent to all three semicircles P, Q, R at the points D, E, and F respectively, as shown in the diagram. Calculate, with proof, the area of triangle DEF.



Credit This problem was contributed by Professor Vladimir Fainzilberg of the Department of Chemistry at the C. W. Post Campus of Long Island University.

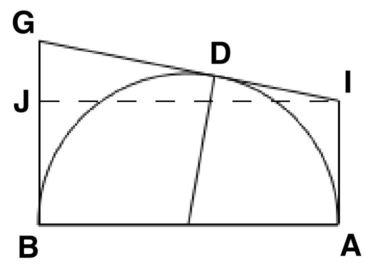
Comments The most common mistake in this problem was asserting that points D and E are at the midpoints of the their respective semi-circles. Some students successfully slogged through this problem with calculus or coordinates. Lawrence Chan shows us a geometric solution and Tony Liu mixes in a little trigonometry. *Solutions edited by Richard Rusczyk*

Solution 1 by: Lawrence Chan (11/IL)



We begin this problem by first drawing lines through A, B, and C perpendicular to the plane of triangle ABC. Then, we draw the lines where the plane containing triangle DEF (that is, the plane tangent to all three circles) intersects each of the three circles' plane (labeling intersection points as shown above). Because $\overline{BG}$ and $\overline{CH}$ were drawn perpendicular to the plane of triangle ABC, and because of symmetry due to the two closest circles being congruent, we know that $BGHC$ is a rectangle. Thus, $BG = CH = 5\sqrt{3}$.

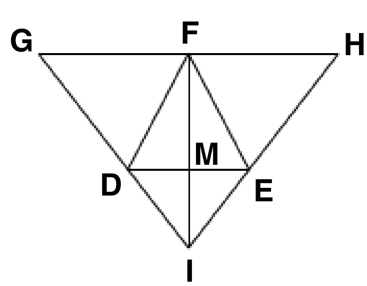
We now turn our focus to the plane of the circle with diameter $\overline{AB}$.



Since $\overline{BG}$, $\overline{GD}$, $\overline{DI}$, and $\overline{IA}$ are all tangent to the semicircle, we know that $BG = GD = 5\sqrt{3}$ and $DI = IA$. Let us call the length of $IA = x$. If we draw a line parallel to $\overline{BA}$ and passing through I, we form a right triangle GJI and a rectangle $BJIA$. Thus, $JB = IA = DI = x$ and $JI = BA$. Using the Pythagorean Theorem on GJI gives us the following result:

$$\begin{aligned} GJ^2 + JI^2 &= GI^2 \\ (GB - JB)^2 + BA^2 &= (GD + DI)^2 \\ (5\sqrt{3} - x)^2 + (10\sqrt{2})^2 &= (5\sqrt{3} + x)^2 \\ 75 - 10x\sqrt{3} + x^2 + 200 &= 75 + 10x\sqrt{3} + x^2 \\ 20x\sqrt{3} &= 200 \\ x &= \frac{10\sqrt{3}}{3} \end{aligned}$$

We now our attention to the plane containing triangles DEF and GHI.



Since the two trapezoids below $\overline{GI}$ and $\overline{HI}$ are congruent and the two circles below the same lines are also congruent, we know that the triangle GHI posses symmetry about $\overline{FI}$.

Thus, we know that DE and GH are parallel, and we can consequently form similar triangles GHI and DEI. We can then set up the following relations.

$$\begin{aligned}
\frac{DE}{GH} &= \frac{ID}{IG} \\
\frac{DE}{GH} &= \frac{x}{x+DG} \\
\frac{DE}{10\sqrt{3}} &= \frac{\frac{10\sqrt{3}}{3}}{\frac{10\sqrt{3}}{3}+5\sqrt{3}} \\
DE &= 4\sqrt{3}
\end{aligned}$$

All that is left is to find FM. We first find the length of FI. Since we have symmetry, we know that GFI is a right triangle, and thus we can use the Pythagorean Theorem.

$$\begin{aligned}
FI &= \sqrt{GI^2 - GF^2} \\
FI &= \sqrt{\left(\frac{10\sqrt{3}}{3}+5\sqrt{3}\right)^2 - \left(5\sqrt{3}\right)^2} \\
FI &= \frac{20\sqrt{3}}{3}
\end{aligned}$$

Now we can use the similar triangles GFI and DMI.

$$\begin{aligned}
\frac{FM}{FI} &= \frac{GD}{GI} \\
\frac{FM}{FI} &= \frac{GD}{GD+x} \\
\frac{FM}{\frac{20\sqrt{3}}{3}} &= \frac{5\sqrt{3}}{5\sqrt{3}+\frac{10\sqrt{3}}{3}} \\
FM &= 4\sqrt{3}
\end{aligned}$$

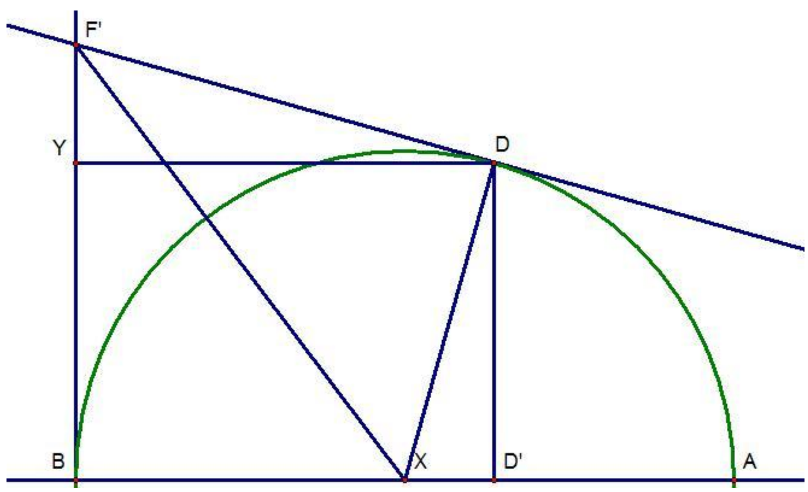
Finally, the area of $DEF = \frac{1}{2}(DE)(FM) = \frac{1}{2}(4\sqrt{3})(4\sqrt{3}) = 24$

$\boxed{Area_{DEF} = 24}$

■

Solution 2 by: Tony Liu (10/IL)

Construct a line through F, parallel to BC, and let it intersect plane BDA at point F'. Since $BC||DE$, F' lies on the plane DEF. Now, let us focus on plane BDA and employ methods of two-dimensional geometry.



Let D' be the foot of the perpendicular from D to AB, and denote the midpoint of AB by X. Additionally, let the foot of the perpendicular from D to BF' be Y. Note that $F'D = F'B$ are tangents to the circle, since F' lies in the planes BFC and DEF. This, along with $BX = DX$ implies that $\triangle F'XB \cong \triangle F'XD$. We note that $F'X = \sqrt{BX^2 + F'B^2} = \sqrt{50 + 75} = 5\sqrt{5}$, since $F'B$ is a radius of the semicircle with diameter $BC = 10\sqrt{3}$. Letting $\theta = \angle F'XB = \angle F'XD$, we have,

$$\cos\theta = \frac{\sqrt{2}}{\sqrt{5}} \Longrightarrow \cos 2\theta = 2\cos^2\theta - 1 = -\frac{1}{5} \Longrightarrow \cos(180 - 2\theta) = \frac{1}{5}$$

and since $\angle DXD' = 180 - 2\theta$, it follows that $XD' = \sqrt{2}$, so $DD' = \sqrt{50 - 2} = 4\sqrt{3}$. In particular, we observe that $\angle BXD$ is obtuse, so D' lies on segment DA. Next, we note that $BYDD'$ is a rectangle (by construction) so $BY = DD'$ and $BD' = YD$. We will use these results later on.

Note that $\triangle DEF$ is isosceles (by symmetry of $\triangle ABC$), and since $AD' = 4\sqrt{2}$, by symmetry and similar isosceles triangles (projecting DE onto $\triangle ABC$), we deduce that $DE = 4\sqrt{3}$. Now, let h denote the altitude of $\triangle DEF$. By using the perpendicular bisector of DE (parallel to plane ABC) and a perpendicular from F to BC, we can calculate h by using the

Pythagorean Theorem. The right triangle has one leg of length $\frac{1 - DE}{BC} \cdot \sqrt{200 - 75} = 3\sqrt{5}$ and another of length $F'Y = \sqrt{75 - 72} = \sqrt{3}$. Thus, we get $h = \sqrt{45 + 3} = 4\sqrt{3}$, and consequently the area of $\triangle DEF$ is $\frac{1}{2} \cdot h \cdot DE = \frac{1}{2} \cdot 4\sqrt{3} \cdot 4\sqrt{3} = 24$.