USA Mathematical Talent Search
Solutions to Problem 1/2/16
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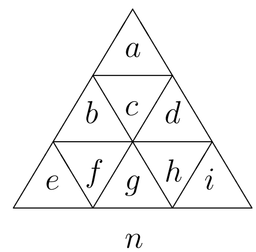
1/2/16. The numbers 1 through 9 can be arranged in the triangles labeled a through i illustrated on the right so that the numbers in each of the 2×2 triangles sum to the same value n; that is

$$a + b + c + d = b + e + f + g = d + g + h + i = n.$$

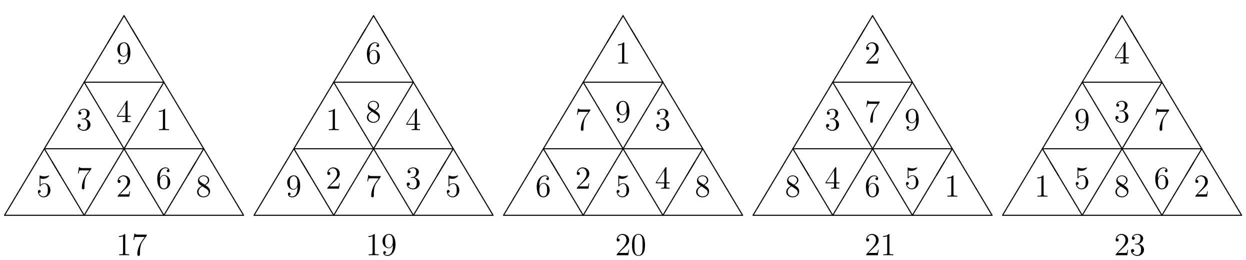


For each possible sum n, show such an arrangement, labeled with the sum as shown at right. Prove that there are no possible arrangements for any other values of n.

Credit This is a take-off on a Hungarian problem that appeared in the book *Brainteasers for Upperclassmen* by Imrecze, Reiman, and Urbán in Hungarian in 1986.

Comments There are basically two steps to the solution: finding bounds on the possible values of n, and then finding which values within those bounds have valid arrangements. Many students simplified the argument by noting the symmetry between arrangements summing to n and arrangements summing to $40 - n$. (Solution 2 below uses this fact.) The major variation amongst different solutions is the method by which the cases $n = 18$ and $n = 22$ were shown to be impossible. Solution 1 shows a nice casework approach. Solution 2 uses a clever observation to eliminate all of the cases at once.

Solution 1 by: Eric Paniagua (12/NY)



From the problem, we have

$$3n = a + b + c + d + e + f + g + h + i + (b + d + g)$$
$$3n = 45 + b + d + g$$

where $\{a, b, c, d, e, f, g, h, i\} = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Clearly, the maximum and minimum values of this sum are

$$\max = 45 + 7 + 8 + 9 = 69$$
$$\min = 45 + 1 + 2 + 3 = 51$$

so we have the bounds on n:

$$51 \leq 3n \leq 69$$
$$17 \leq n \leq 23$$

Example arrangements for $n = 17, 19, 20, 21, 23$ are given above.

Proof that no arrangement exists for $n = 18$.
If $n = 18$ then $b+d+g = 3(18)-45 = 9$ and $\{b, d, g\}$ equals exactly one of $\{4, 2, 3\}, \{1, 5, 3\}, \{1, 2, 6\}$. By the symmetry of the positions of b, d, g in the triangle all assignments of these to the numbers in one of these sets are equivalent.

First, assume $b = 4, d = 2, g = 3$. Then we have

$$n = 18 = a + b + c + d = a + c + 6$$

and $a + c = 12$, so $\{a, c\} = \{7, 5\}$ because this is the only decomposition of 12 not using 2, 3, or 4. Similarly, $e + f = 18 - b - g = 11$ and $\{e, f\} = \{6, 5\}$ contradicting the fact that a, c, e, f are distinct.

Now assume $b = 1, d = 5, g = 3$. We have $a + c = 18 - b - d = 12$, so $\{a, c\} = \{8, 4\}$. Similarly, $h + i = 18 - d - g = 10$ and $\{h, i\}$ equals $\{6, 4\}$ or $\{8, 2\}$ contradicting the fact that a, c, h, i are distinct.

Finally, assume $b = 1, d = 2, g = 6$. Then $a + c = 18 - b - d = 15$, so $\{a, c\} = \{8, 7\}$. Similarly, $e + f = 18 - b - g = 11$ and $\{e, f\}$ equals $\{7, 4\}$ or $\{8, 3\}$ contradicting the fact that a, c, e, f are distinct.

$\therefore n \neq 18$. □

Proof that no arrangement exists for $n = 22$.
If $n = 22$ then $b+d+g = 3(22)-45 = 21$ and $\{b, d, g\}$ equals exactly one of $\{6, 7, 8\}, \{9, 7, 5\}, \{9, 4, 8\}$.

Assume $b = 6, d = 7, g = 8$. Then $a + c = 22 - b - d = 9$, so $\{a, c\} = \{5, 4\}$. Similarly, $e + f = 22 - b - g = 8$ and $\{e, f\} = \{5, 3\}$ contradicting the fact that a, c, e, f are distinct.

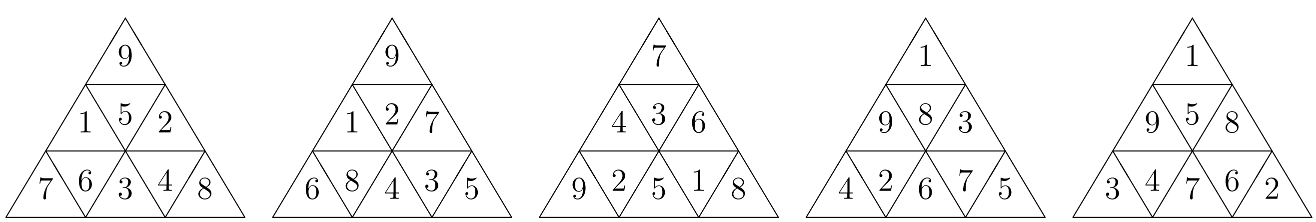
Assume $b = 9, d = 7, g = 5$. Then $a + c = 22 - b - d = 6$, so $\{a, c\} = \{4, 2\}$. Similarly, $e + f = 22 - b - g = 8$ and $\{e, f\} = \{6, 2\}$ contradicting the fact that a, c, e, f are distinct.

Assume $b = 9, d = 4, g = 8$. Then $a + c = 22 - b - d = 9$, so $\{a, c\}$ equals $\{7, 2\}$ or $\{6, 3\}$. Similarly, $e + f = 22 - b - g$ and $\{e, f\} = \{3, 2\}$ contradicting the fact that a, c, e, f are distinct.

$\therefore n \neq 22$. □

Solution 2 by: Adam Hesterberg (10/WA)

Answer: The possible values of n are 17, 19, 20, 21, and 23, as shown below:



17 19 20 21 23

First, we prove that $17 \leq n \leq 23$. Note that

$$\begin{aligned}3n &= (a+b+c+d)+(b+e+f+g)+(d+g+h+i)\\ &= \sum(\text{all the numbers})+b+d+g\\ &= 45+b+d+g\end{aligned}$$

Since $b+d+g$ is between $1+2+3=6$ and $7+8+9=24$, n is between $\frac{45+6}{3}=17$ and $\frac{45+24}{3}=23$.

Thus, the only values for n left to consider are 18 and 22. Note that if 18 could be constructed, so could 22, by replacing each entry x by $10-x$. Therefore, we need only consider 18.
$n=18$ implies $b+d+g=3*18-45=9$. Without loss of generality, let 9 be a (it will not be b, d, or g since the sum of the other two would then have to be 0). Then,

$$\begin{aligned}18 &= a+b+c+d\\ &= 9+c+(b+d+g)-g\\ &= 9+9+c-g\\ 0 &= c-g\\ c &= g\end{aligned}$$

However, c and g were to be distinct, so this is impossible. Therefore, neither 18 nor 22 can be constructed, so the only possible values for n are the ones constructed above.

2/2/16. Call a number $a - b\sqrt{2}$ with a and b both positive integers *tiny* if it is closer to zero than any number $c - d\sqrt{2}$ such that c and d are positive integers with $c < a$ and $d < b$. Three numbers which are tiny are $1 - \sqrt{2}$, $3 - 2\sqrt{2}$, and $7 - 5\sqrt{2}$. Without using a calculator or computer, prove whether or not each of the following is tiny:

$$(a)\ 58 - 41\sqrt{2}\,, \qquad (b)\ 99 - 70\sqrt{2}\,.$$

Credit We are indebted to Dr. David Grabiner of the NSA for this problem. David is a former multiple winner of the USAMO, whose continued support of the USAMTS is most appreciated.

Comments Solution 1 shows the most straightforward solution. Solution 2 uses the shape of the graph of $y = \sqrt{x}$. Solution 3 uses the continued fraction representation of $\sqrt{2}$. Other solutions are possible, including listing (by hand!) all of the smallest numbers of the form $|a - b\sqrt{2}|$ for each positive integer a up through 100.

Solution 1 by: Tony Liu (10/IL)

(a) We claim that $58 - 41\sqrt{2}$ is not tiny. Indeed, from $1 < \sqrt{2}$, we have

$$\begin{aligned} |58 - 41\sqrt{2}| &> \frac{|58 - 41\sqrt{2}|}{\sqrt{2}} \\ &= |29\sqrt{2} - 41| \\ &= |41 - 29\sqrt{2}| \end{aligned}$$

Thus $41 - 29\sqrt{2}$ is closer to zero than $58 - 41\sqrt{2}$. Since $41 < 58$, and $29 < 41$, we conclude that $58 - 41\sqrt{2}$ is not tiny.

(b) We claim that $99-70\sqrt{2}$ is tiny. Assume, for the sake of contradiction, that there exists a number $c-d\sqrt{2}$ closer to zero, with $c < 99$, and $d < 70$. Since $99^2-2\cdot 70^2 = 9801-9800 = 1$, we have

$$\begin{aligned} 1 &= |(99 - 70\sqrt{2})(99 + 70\sqrt{2})| \\ &> |(c - d\sqrt{2})(99 + 70\sqrt{2})| \\ &> |(c - d\sqrt{2})(c + d\sqrt{2})| \\ &= |c^2 - 2d^2| \end{aligned}$$

Because c and d are positive integers, this implies that $c^2 - 2d^2 = 0$, or $c = d\sqrt{2}$, which is impossible. It follows that $99 - 70\sqrt{2}$ is indeed tiny.

Solution 2 by: Johnny Hu (10/AL)

The three examples for *tiny* numbers are all in the form, $\sqrt{x+1} - \sqrt{x}$ or $\sqrt{x} - \sqrt{x+1}$, where x is an integer. Since the graph for $\sqrt{x}$ is half a parabola that opens to the positive side that rises more and more slowly as x increases, the difference between $\sqrt{x+1}$ and $\sqrt{x}$ becomes smaller and smaller as x increases. Since $x+1$ and x are consecutive integers and the difference between $\sqrt{x+1}$ and $\sqrt{x}$ becomes smaller as x increases, numbers in the form of $\sqrt{x+1} - \sqrt{x}$ and $\sqrt{x} - \sqrt{x+1}$ must be *tiny* because all values smaller than x will not produce a number closer to zero.

Since $58 - 41\sqrt{2}$ can be written as $\sqrt{3364} - \sqrt{3362}$, it is in the form of $\sqrt{x+2} - \sqrt{x}$. The graph of $\sqrt{x+2} - \sqrt{x}$ is above the graph of $\sqrt{x+1} - \sqrt{x}$ so $58 - 41\sqrt{2}$ is not a tiny number as there exists a number in the form of $c - d\sqrt{2}$, where $c < 58$ and $d < 41$, which is closer to zero.
To verify this, we must find a number in the form of $\sqrt{y+1} - \sqrt{y}$ or $\sqrt{y} - \sqrt{y+1}$, since these will most likely to be smaller than $\sqrt{x+2} - \sqrt{x}$ (This will be proven later in the page).
$a - b\sqrt{2} = 58 - 41\sqrt{2}$

$$\begin{aligned} 58 - 41\sqrt{2} &= a - b\sqrt{2} \\ &= \sqrt{a^2} - \sqrt{2b^2} \end{aligned}$$

Also:

$$\begin{aligned} a^2 &= 2b^2 + 2 \\ &= 2(b^2 + 1) \end{aligned}$$

To make this equation into the form of $\sqrt{y} - \sqrt{y+1}$:
Let:

$$(d)(\sqrt{2}) = a$$

Then:

$$2d^2 = 2(b^2 + 1)$$

$$d^2 = b^2 + 1$$

$$b^2 = y$$

$$d^2 = y + 1$$

Therefore, $\sqrt{y} - \sqrt{y+1} = \sqrt{b^2} - \sqrt{d^2}$
Substituting our original value, we have $\sqrt{1681} - \sqrt{1682} = 41 - 29\sqrt{2}$.
To verify that $41 - 29\sqrt{2}$ is closer to zero than $58 - 41\sqrt{2}$:

$$|41 - 29\sqrt{2}| < |58 - 41\sqrt{2}|$$

$$|41 - 29\sqrt{2}|^2 < |58 - 41\sqrt{2}|^2$$

$$3363 - 2378\sqrt{2} < 6726 - 4756\sqrt{2}$$

Since $2(3363 - 2378\sqrt{2}) = 6726 - 4756\sqrt{2}$, $|41 - 29\sqrt{2}| < |58 - 41\sqrt{2}|$ and $58 - 41\sqrt{2}$ is not a *tiny* number.

Since $99-70\sqrt{2}$ can be written as $\sqrt{9801}-\sqrt{9800}$, it is in the form of $\sqrt{x+1}-\sqrt{x}$. Numbers in this form are always *tiny* numbers, so $99 - 70\sqrt{2}$ is a *tiny* number.

Solution 3 by: Zachary Abel (11/TX)

This problem follows from a (well known?) theorem concerning the approximation ability of continued fractions.

Theorem. *For a given irrational number α, the number $p - q\alpha$ is* tiny *if and only if p/q is a convergent of α.*

The proof is in two parts.

Lemma 1. *If p_n/q_n is a convergent for the irrational number α and $p/q \neq p_n/q_n$ is an arbitrary fraction with $0 < q < q_{n+1}$, then*

$$|p_n - q_n\alpha| < |p - q\alpha|.$$

Proof. The key to this proof is to try to write

$$(p_n - q_n\alpha)x + (p_{n+1} - q_{n+1}\alpha)y = p - q\alpha$$

by solving the system

$$\begin{cases} q_n x + q_{n+1} y &= q \\ p_n x + p_{n+1} y &= p \end{cases} \qquad (1)$$

for x and y. Using the fact that $p_{n+1}q_n - p_n q_{n+1} = (-1)^n$, we find from the above system that

$$x = (-1)^n\big(qp_{n+1} - pq_{n+1}\big) \quad \text{and} \quad y = (-1)^n\big(pq_n - qp_n\big)$$

This tells us a lot! First of all, both x and y are integers. Next, neither x nor y is 0. Indeed, if $x = 0$ then $p/q = p_{n+1}/q_{n+1}$, which is impossible for $q < q_{n+1}$ since $\gcd(p_{n+1}, q_{n+1}) = 1$, and if $y = 0$ then $p/q = p_n/q_n$, which was assumed to be false.

We can obtain even more information from the sytem in (1): x and y must have opposite sign. If both were positive, then $q = q_n x + q_{n+1} y > q_{n+1}$, and if both were negative, then q would be negative.

Now we're ready for the final step. Since α lies between p_n/q_n and p_{n+1}/q_{n+1}, the numbers $p_n - q_n\alpha$ and $p_{n+1} - q_{n+1}\alpha$ have opposite signs. Since x and y also have opposite signs, the two numbers $(p_n - q_n\alpha)x$ and $(p_{n+1} - q_{n+1}\alpha)y$ have the same sign. Thus,

$$\begin{aligned}
(p_n - q_n\alpha)x + (p_{n+1} - q_{n+1}\alpha)y &= p - q\alpha \\
\big|(p_n - q_n\alpha)x + (p_{n+1} - q_{n+1}\alpha)y\big| &= \big|p - q\alpha\big| \\
\big|(p_n - q_n\alpha)x\big| + \big|(p_{n+1} - q_{n+1}\alpha)y\big| &= \big|p - q\alpha\big| \\
\big|p_n - q_n\alpha\big| \cdot \big|x\big| &< \big|p - q\alpha\big| \\
\big|p_n - q_n\alpha\big| &< \big|p - q\alpha\big|
\end{aligned}$$

□

Notice that this lemma shows that the number $p_n - q_n\alpha$ is *tiny*. This next lemma shows that there are no other tiny numbers.

Lemma 2. *If p/q is not a convergent of α, then $p - q\alpha$ is not tiny.*

Proof. Since p/q isn't a convergent to α, we can find two successive convergents p_n/q_n and p_{n+1}/q_{n+1} with $q_n < q < q_{n+1}$. Then by the first lemma, $|p_n - q_n\alpha| < |p - q\alpha|$, and so $p - q\alpha$ is not *tiny*. □

These two lemmas show that $p_n - q_n\alpha$ is tiny for each n and that there are no other tiny numbers. So the main theorem has been proven.

Because of this theorem with $\alpha = \sqrt{2}$, the tiny numbers can be found by calculating the convergents of $\sqrt{2}$. The continued fraction representation of $\sqrt{2}$ is

$$\sqrt{2} = 1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}}}$$

Using the recurrence relations

$$\begin{aligned}
p_n &= a_n p_{n-1} + p_{n-2} \\
q_n &= a_n q_{n-1} + q_{n-2}
\end{aligned}$$

and the fact that $a_n = 2$ for $n \geq 1$, we can easily calculate the convergents. We get

$$\frac{p_0}{q_0} = \frac{1}{1} \qquad \frac{p_1}{q_1} = \frac{3}{2} \qquad \frac{p_2}{q_2} = \frac{7}{5} \qquad \frac{p_3}{q_3} = \frac{17}{12}$$
$$\frac{p_4}{q_4} = \frac{41}{29} \qquad \frac{p_5}{q_5} = \frac{99}{70} \qquad \frac{p_6}{q_6} = \frac{239}{169} \qquad \frac{p_7}{q_7} = \frac{577}{408}$$

Since $99/70$ is one of the convergents, $99 - 70\sqrt{2}$ is a tiny number, whereas $58/41$ is not a convergent and so $58 - 41\sqrt{2}$ is not tiny.

3/2/16. A set is *reciprocally whole* if its elements are distinct integers greater than 1 and the sum of the reciprocals of all those elements is exactly 1. Find a set S, as small as possible, that contains two reciprocally whole subsets, I and J, which are distinct but not necessarily disjoint (meaning they may share elements, but they may not be the same subset). Prove that no set with fewer elements than S can contain two reciprocally whole subsets.

Credit We are thankful to Dr. Kent D. Boklan of the National Security Agency for devising this nice problem.

Comments There are many possible 5-element sets which satisfy the conditions of the problem; probably the one most commonly cited in student's solutions was $\{2, 3, 4, 6, 12\}$. The key to this problem was *rigorously* proving that a set with 4 or fewer elements is impossible. Solution 1 is an especially concise example. Some solutions, such as Solution 2, prove along the way that $\{2, 3, 6\}$ is the unique reciprocally whole set with 3 elements.

Solution 1 by: Zhou Fan (11/NJ)

A reciprocally whole set must have at least three elements, since the reciprocals of only two distinct integers greater than 1 can sum to at most $1/2 + 1/3 = 5/6$. Thus a set S with two reciprocally whole subsets must contain at least four elements. Suppose such a set exists: $S = \{a, b, c, d\}$. The entire four element set cannot be reciprocally whole if there is a smaller reciprocally whole subset, so S must contain two three-element reciprocally whole subsets. WLOG, assume that they are $\{a, b, c\}$ and $\{a, b, d\}$. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1$ and $\frac{1}{a} + \frac{1}{b} + \frac{1}{d} = 1$, which implies that $\frac{1}{c} = \frac{1}{d}$, or $c = d$: a contradiction. Thus S has at least 5 elements. Such a five-element set exists: $S = \{2, 3, 6, 7, 42\}$ satisfies the problem conditions since $1/2 + 1/3 + 1/6 = 1$ and $1/2 + 1/3 + 1/7 + 1/42 = 1$.

Solution 2 by: Jason Ferguson (12/TX)

Consider the set $S = \{2, 3, 6, 9, 18\}$. S is a 5-element set that contains two distinct reciprocally whole subsets, $\{2, 3, 6\}$ and $\{2, 3, 9, 18\}$ (since $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{18} = 1$). We will now show that there are no sets with two distinct reciprocally whole subsets with cardinality 0, 1, 2, 3, or 4.

To this end, we first show that there cannot be a reciprocally whole set of cardinality zero, one, or two and the only reciprocally whole set of cardinality three is $\{2, 3, 6\}$. Clearly the null set cannot be reciprocally whole, and if the reciprocal of a number is 1, then that number must be one. Thus, there is only 1 one-element set S which has the property that the sum of the reciprocals of its elements, and that set is $\{1\}$, but this is not a reciprocally whole set (all elements must be greater than 1). Suppose T is a reciprocally whole set of cardinality two, and let x be the smaller element and y the larger (the two elements are distinct). Then $\{x, y\}$ is a reciprocally whole set, so $\frac{1}{x} + \frac{1}{y} = 1$. However, because $x < y$, $\frac{1}{x} > \frac{1}{y}$, so $\frac{1}{x} > \frac{1}{2}$. Then $x < 2$. But all elements in a reciprocally whole set must be integers greater than one.

From this contradiction we conclude that there are no 2-element reciprocally whole sets.

Suppose now that U is a three-element reciprocally whole set. Then, let a be the smallest element, b the middle element, and c the largest. Then $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$, and $a<b<c$. Therefore, $\frac{1}{a}>\frac{1}{b}>\frac{1}{c}$, so $\frac{1}{a}>\frac{1}{3}$. Then $a<3$, so $a=2$. Because $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=1$ and $a=2$, it follows that $\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$. Because $\frac{1}{b}>\frac{1}{c}$, it follows that $\frac{1}{b}>\frac{1}{4}$. Then $b<4$. Since $2=a<b$, it follows that $b=3$. Because $\frac{1}{b}+\frac{1}{c}=\frac{1}{2}$ and $b=3$, it follows that $\frac{1}{c}=\frac{1}{2}-\frac{1}{3}=\frac{1}{6}$. Then, $c=6$, so $(a,b,c)=(2,3,6)$, and we conclude that the only reciprocally whole three-element set is $\{2,3,6\}$. Therefore, there is only one reciprocally whole set with cardinality less than or equal to 3: $\{2,3,6\}$.

A set with cardinality of 0 or 1 cannot have two distinct, nonempty subsets. Since a reciprocally whole set has to be nonempty, it follows that a set of cardinality 0 or 1 cannot have two distinct reciprocally whole subsets. Now, if a set V has two distinct reciprocally whole subsets, then neither of those two subsets can be V itself, for if V was reciprocally whole, then the other reciprocally whole subset of V must be a proper subset of V. Then the sum of the reciprocals of the elements of the proper subset would be less than the sum of the reciprocals of the elements of V, which is 1. From this contradiction we conclude that if a set V has two reciprocally whole subsets, then both of them must be proper subsets of V. Thus, if a set with cardinality 2 had two distinct, reciprocally whole subsets, then both of them would have to have cardinality less than or equal to 1. This is impossible, as there are no reciprocally whole sets of cardinality 0 or 1. Similarly, because there are also no reciprocally whole sets of cardinality 2, a set with cardinality 3 cannot have two distinct reciprocally whole subsets. Finally, if a set with cardinality 4 had two distinct, reciprocally whole subsets, then both of them would have to have cardinality less than or equal to 3. This is impossible, as there is only one reciprocally whole set of cardinality less than or equal to 1.

So we conclude that there are no sets with two distinct, reciprocally whole subsets whose cardinality is less than or equal to 4, but the set $S=\boxed{\{2,3,6,9,18\}}$ is a five-element set with this property. We also conclude that S is a minimal set with two distinct, reciprocally whole subsets. QED

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Solutions to Problem 4/2/16
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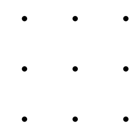
4/2/16. How many quadrilaterals in the plane have four of the nine points $(0,0)$, $(0,1)$, $(0,2)$, $(1,0)$, $(1,1)$, $(1,2)$, $(2,0)$, $(2,1)$, $(2,2)$ as vertices? Do count both concave and convex quadrilaterals, but do not count figures where two sides cross each other or where a vertex angle is 180°. Rigorously verify that no quadrilateral was skipped or counted more than once.



Credit This problem was proposed by Professor Harold Reiter, the president of Mu Alpha Theta Mathematics Honor Society and a long-time supporter of the USAMTS program.

Comments Though a combinatorial argument is the easiest place to start with this problem, many students found interesting ways to show that they covered all possible cases. Many of those methods failed due to the difficulty of making sure every case was covered. Most such methods would scale poorly to problems with larger grids.

Solution 1 by: Derrick Sund (11/NC)
There are $\binom{9}{4} = 126$ different ways to choose four of the points to be the vertices of a quadrilateral. However, from this we must subtract the number of ways there are to choose those four points such that three of them are colinear. There are 8 ways to choose three points such that all of them are colinear, and for all of them, there are 6 ways to choose the fourth point, so the number of ways to choose four points such that they can form a quadrilateral that actually has four distinct sides is $\binom{9}{4} - 6 \times 8 = 78$.

We are not done yet. Some of those sets of four points will give us exactly one quadrilateral (such as $(0,0),(0,1),(1,1),(1,0)$), while others (such as $(0,0),(1,2),(1,1),(2,1)$) give us three. A set of four points with no three of the points colinear will give us three quadrilaterals if one of the points is inside the triangle formed by the other three; otherwise, the four points give us one quadrilateral. There are exactly eight sets of points such that one is inside the triangle formed by the other three:

$$(1,1),(0,2),(1,0),(2,1)$$

$$(1,1),(2,2),(1,0),(0,1)$$

$$(1,1),(2,0),(0,1),(1,2)$$

$$(1,1),(0,0),(2,1),(1,2)$$

$$(1,1),(1,2),(0,0),(2,0)$$

$$(1,1),(2,1),(0,0),(0,2)$$

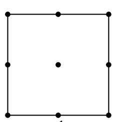
$$(1,1),(1,0),(0,2),(2,2)$$

$$(1,1),(0,1),(2,0),(2,2)$$

The other 70 possible sets of points all give us one quadrilateral. Therefore, the answer to the problem is $1(70) + 3(8) = 94$.

Solution 2 by: Keone Hon (10/HI)
There are **94** such quadrilaterals. We will separate the quadrilaterals into a number of cases as a primary means of enumerating them:

Rectangles: There are 10 rectangles in all: one 2×2, one $\sqrt{2} \times \sqrt{2}$, four 1×1s, and four 1×2s. An example of each is shown below: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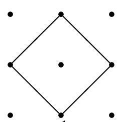
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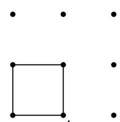
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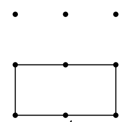
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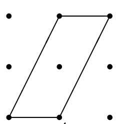
Parallelograms (other than rectangles): there are two types of parallelograms with non-perpendicular sides: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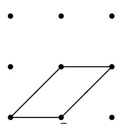
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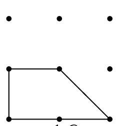
There are four parallelograms of the first shape, since the pair of currently horizontal sides may be any of the four pairs of opposite sides on an edge of the figure. There are eight parallelograms of the second shape, since for each of the four 1×2 rectangles, there are two such parallelograms inside. In all, there are then 12 non-rectangle parallelograms.

Trapezoids: There are three types of trapezoids: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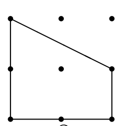
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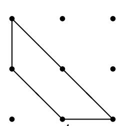
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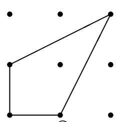
There are 16 trapezoids of the first type, since each of the four 1×2 rectangles contains four such trapezoids (each one is formed by cutting off a corner, and there are four corners). There are 8 trapezoids of the second type, since any of the four edges of the array can be the side that is currently on the bottom, and for each of those four choices, the longer of the two bases can be chosen from among two bases. There are four trapezoids of the third type, since the two non-parallel edges can (when extended) meet at any of the four corners of the array. In all, there are 28 trapezoids.

Kites: There is only one type of kite. There are four of this type, since the right-angle vertex can be chosen from any of the four corners of the array.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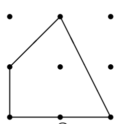
Other convex figures: There are two other types of convex figures with no parallel sides. For the first shape, any of the four edges of the array can be chosen as the side with length 2, and from there, any of the two adjacent edges can be chosen as the side with length 1. Thus there are 8 of this shape. For the second shape, any of the 8 segments on the edge of the array with length 1 can be chosen as the side with length 1; from there, the rest of the shape is determined. Thus there are 8 of this shape. In all, then, there are 16 such figures.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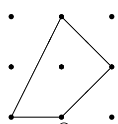
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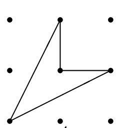
Concave figures: There are four types of convex figures. There are four of the first shape, since any of the four corners of the array can be chosen as the vertex currently in the bottom left corner. There are eight of the second shape, since in addition to the same four choices for the corner vertex, there are also two choices for which side the obtuse angle will open towards. There are eight of the third shape, since there are four choices for the side of length 2 (it can lie on any of the four edges of the array) and two choices for which side the obtuse angle will open towards. There are four of the last shape, since the obtuse angle can open towards any of the four edges of the array. In all, there are 24 concave figures.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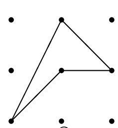
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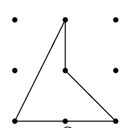
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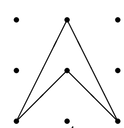
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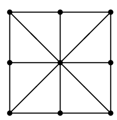
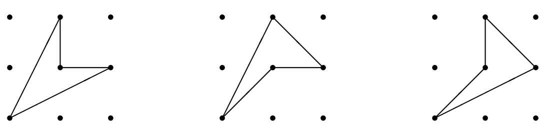
These are all the shapes. In all, there are $10 + 12 + 28 + 4 + 16 + 24 =$ **94 quadrilaterals**.

To verify that all the shapes have been counted exactly once, consider the following. There are $\binom{9}{4} = 126$ ways to choose 4 points out of the 9. Of all combinations of four points, the only ones that do not form quadrilaterals are where three points are collinear. Three points can be collinear if they are on any of the eight lines passing through three points, as shown below. The fourth point can be chosen from any of the other 6 points not on that line. Thus, there are 48 combinations of four points that cannot form quadrilaterals. So there are $126 - 48 = 78$ combinations of four points that do form quadrilaterals.

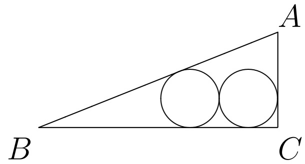


Given any collection of four noncollinear points, exactly one convex figure can be drawn through them. Thus, 70 combinations of four points correspond to 70 convex quadrilaterals. However, this is not true for concave figures; more than one concave figure can be drawn through four points. Each of the remaining eight combinations can be formed into three distinct quadrilaterals, as demonstrated below. Thus there are 24 concave quadrilaterals. In all, there are $70 + 24 =$ **94 quadrilaterals**, which agrees with our previous count.

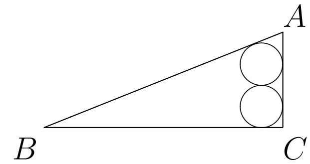


5/2/16.

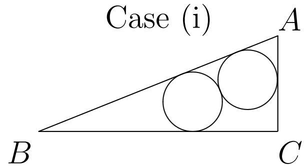
Two circles of equal radius can tightly fit inside right triangle ABC, which has $AB = 13$, $BC = 12$, and $CA = 5$, in the three positions illustrated below. Determine the radii of the circles in each case.



Case (i)



Case (ii)



Case (iii)

Credit This problem was inspired by Problem 5.3.2 in *Traditional Japanese Mathematics Problems of the 18th and 19th Centuries* published in 2002 by SCT Publishing of Singapore. We are thankful to Mr. Willie Yong, the Publisher, for sending us a copy of this wonderful book.

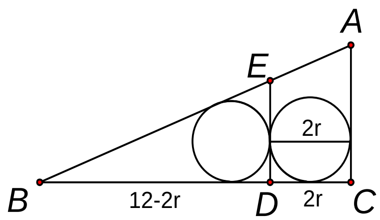
Comments There are many ways to solve this problem though most of them used right angles drawn from the centers of the circles to points of tangency and the use of similar triangles (whether or not they were implied by the use of trigonometry).

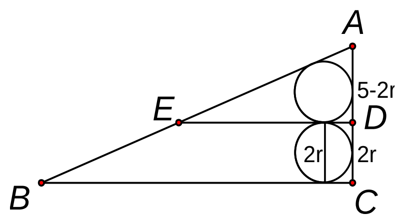
Solution 1 by: Jeffrey Manning (9/CA)

The method for solving case (iii) is different than the method for solving cases (i) and (ii) so we will look at these seperatly.

Cases (i) and (ii)

Draw the line through the point of intersection of the two circles that is tangent to both circles. Call the point where this line intersects the hypotenuse, E, and call the point where it intersects the leg that is tangent to both circles (BC in case (i) and CA in case (ii)), D.





In each case ED is perpendicular to one leg and parallel to the other. This means that in case (i) $\triangle ABC \sim \triangle EBD$ and in case (ii) $\triangle ABC \sim \triangle AED$. In each of these cases one of the circles is the incircle of the new triangle. Also because in each case the distance from ED to the leg it is parallel to is $2r$ (where r is the length of the radii of the circles) we have $DC = 2r$

The length of the radius, R, of the incircle of a right triangle is given by $R = \dfrac{ab}{a+b+c}$. Since $a = 5$, $b = 12$ and $c = 13$, we have $R = 2$. R will be proportional to r.

We will now solve each case seperatly.

Case (i)

Since $DC = 2r$ we have $BD = 12 - 2r$ so, since $\triangle ABC \sim \triangle EBD$,

$$\begin{aligned} \frac{BD}{BC} &= \frac{r}{R} \\ \frac{12-2r}{12} &= \frac{r}{2} \\ 24 - 4r &= 12r \\ 16r &= 24 \\ \mathbf{r} &= \mathbf{3/2} \end{aligned}$$

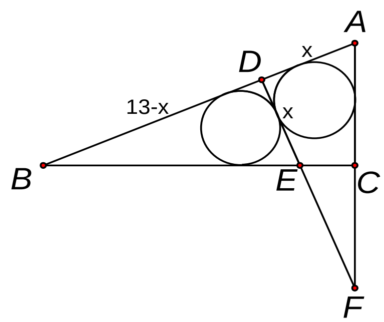
Case (ii)

Since $DC = 2r$ we have $AD = 5 - 2r$ so, since $\triangle ABC \sim \triangle EBD$,

$$\begin{aligned} \frac{AD}{AC} &= \frac{r}{R} \\ \frac{5-2r}{5} &= \frac{r}{2} \\ 10 - 4r &= 5r \\ 9r &= 10 \\ \mathbf{r} &= \mathbf{10/9} \end{aligned}$$

Case (iii)

Draw the line tangent to both circles similarly to cases (i) and (ii). Call the points where it intersects AB, BC and the line formed by AC, D, E and F respectively.



$\triangle EBD$ and $\triangle AFD$ are both right triangles because $AB \perp DF$. Also $\angle ABC = \angle EBD$ and $\angle BAC = \angle FAD$, so $\triangle EBD \sim \triangle ABC \sim \triangle AFD$. Since $\triangle EBD \sim \triangle AFD$ and the radii of their incircles are equal we have $\triangle EBD \cong \triangle AFD$, so $AD = DE$.

Let $x = AD = DE$

Since $AD = x$, $DB = 13 - x$ which gives:

$$\frac{x}{CA} = \frac{13-x}{BC}$$

$$\begin{aligned} 12x &= 65 - 5x \\ 17x &= 65 \\ x &= 65/17 \end{aligned}$$

$$\frac{r}{R} = \frac{x}{CA}$$

$$\frac{r}{2} = \frac{x}{5}$$

$$r = \frac{2}{5}x$$

$$r = \left(\frac{2}{5}\right)\left(\frac{65}{17}\right)$$

$$\mathbf{r} = \mathbf{26/17}$$

This can be generalized for $AB = c$, $BC = a$ and $CA = b$:

Case (i)

We have $BD = a - 2r$ so

$$\mathbf{r} = \frac{\mathbf{aR}}{\mathbf{a + 2r}}$$

Substituting $\frac{ab}{a+b+c}$ for R gives:

$$\mathbf{r} = \frac{\mathbf{ab}}{\mathbf{a + 3b + c}}$$

Similarly in case (ii) $\mathbf{r} = \frac{\mathbf{ab}}{\mathbf{3a + b + c}}$

Case (iii)

Since $AD = x$ we have $DB = c - x$ so

$$\frac{x}{b} = \frac{c-x}{a}$$

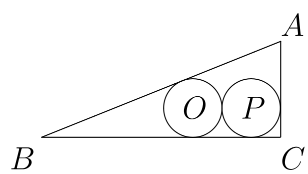
$$x = \frac{bc}{a+b}$$

So

$$r = \frac{R}{b}x = \frac{Rc}{a+b}$$

$$\mathbf{r} = \frac{\mathbf{abc}}{\mathbf{(a+b+c)(a+b)}}$$

Solution 2 by: Chenyu Feng (12/IL)
Case (i)

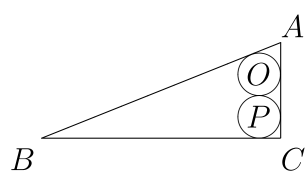


Call the centers of the two circles O and P, with P being closer to point C. Because the area of the triangle is $5 \cdot 12/2 = 30$, we will sum the partial areas to find r.

$$\begin{aligned} K(\text{trapezoid } BCPO) + K(\triangle AOB) + K(\triangle AOP) + K(\triangle APC) &= K(\triangle ABC) \\ \frac{r}{2} \cdot (2r+12) + \frac{13r}{2} + \frac{2r(5-r)}{2} + \frac{5r}{2} &= 30 \\ r &= 3/2 \end{aligned}$$

Thus the radii of both circles in case (i) is $\boxed{3/2}$.

Case (ii)



Call the centers of the two circles O and P, with P being closer to point C. Because the area of the triangle is $5 \cdot 12/2 = 30$, we will again sum the partial areas to find r.

$$\begin{aligned} K(\text{trapezoid } AOPC) + K(\triangle BOA) + K(\triangle BOP) + K(\triangle CPB) &= K(\triangle ABC) \\ \frac{r}{2} \cdot (2r+5) + \frac{13r}{2} + \frac{2r(12-r)}{2} + \frac{12r}{2} &= 30 \\ r &= 10/9 \end{aligned}$$

Thus the radii of both circles in case (ii) is $\boxed{10/9}$.

Case (iii)

We will use similar triangles to solve this one. Firstly, draw the two circles' common internal tangent, intersecting line AB at D and line BC at E. Extend lines DE and AC so that they meet at point F. Because $\angle BDE = \angle ACE$ (they're both right angles), $\angle DEB = \angle A$ and the two incircles are the same size, $\triangle ADF \cong \triangle BDE$. It immediately follows that $AD = DE$. Because $\triangle BDE$ and $\triangle ABC$ each have a right angle and they share $\angle B$, it follows that $\triangle BDE \sim \triangle ABC$. Then, we set up a ratio:

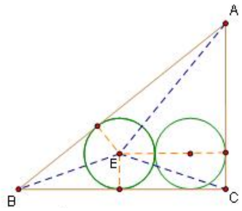
$$\begin{aligned}
\frac{DE}{AC} &= \frac{BD}{BC} \\
\frac{DE}{5} &= \frac{13 - DE}{12} \\
DE &= \frac{65}{17} \\
BD &= 13 - DA = 13 - DE = \frac{156}{17} \\
BE &= \sqrt{DE^2 + BD^2} = \sqrt{\left(\frac{65}{17}\right)^2 + \left(\frac{156}{17}\right)^2} = \frac{169}{17}
\end{aligned}$$

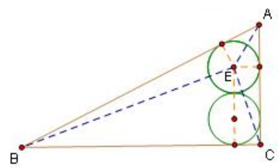
And since in a right triangle, $2r = a + b - c$, where a and b are the lengths of the legs and c is the length of the hypotenuse:

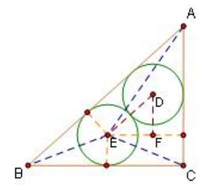
$$\begin{aligned}
2r &= a + b - c \\
r &= \frac{a + b - c}{2} \\
r &= \frac{\frac{156}{17} + \frac{65}{17} - \frac{169}{17}}{2} \\
r &= \frac{26}{17}
\end{aligned}$$

And thus the radius of the circles, r, is $\boxed{26/17}$.

Solution 3 by: Benjamin Lee (9/MD)

Case (i) 

Case (ii) 

Case (iii) 

In all cases, $\text{Area}_{\text{3triangles}} = \text{Area}_{\triangle ABC} = \frac{1}{2}(BC)(AC) = \frac{1}{2}(12)(5) = 30$

In all cases, we use the formula for a triangle area for the 3 triangles ($\triangle EBA$, $\triangle EAC$, $\triangle ECB$) and $\triangle ABC$, and r, b, h, for the radius, base, and height of the small triangle, respectively, so we have

$$\frac{1}{2}BC * h_{BC} + \frac{1}{2}AC * h_{AC} + \frac{1}{2}AB * h_{AB} = 30$$

$$6h_{BC} + \frac{5}{2}h_{AC} + \frac{13}{2}h_{AB} = 30.$$

Case (i)

Drawing auxillary lines (orange) from the center of the leftmost circle to the vertices, we have three small triangles whose area add up to the area of $\triangle ABC$. Thus,

Using the values $h_{BC} = r$, $h_{AC} = 3r$, $h_{AB} = r$, we get

$$6r + \frac{5}{2}(3r) + \frac{13}{2}r = 30$$

$$20r = 30$$

$$r = \frac{3}{2}$$

Case (ii)

Drawing auxillary lines (orange) from the center of the topmost circle to the vertices, we have three small triangles whose area add up to the area of $\triangle ABC$. Thus,

Using the values $h_{BC} = 3r$, $h_{AC} = r$, $h_{AB} = r$, we get

$6(3r) + \frac{5}{2}r + \frac{13}{2}r = 30$

$27r = 30$

$r = \frac{10}{9}$

Case (iii)

Drawing auxillary lines (orange) from the center of the leftmost circle to the vertices, we have three small triangles whose area add up to the area of $\triangle ABC$.

To find the distance from the center of the leftmost circle to AC, because $\triangle ABC \sim \triangle DEF$ we can use the proportion

$$\frac{h_{AC} - r}{BC} = \frac{DE}{AB} = \frac{2r}{AB}.$$

From this proportion,

$$h_{AC} = \frac{24}{13}r + r.$$

Using the values $h_{BC} = r$, $h_{AC} = \frac{24}{13}r + r$, $h_{AB} = r$, we get

$$6r + \frac{13}{2}r + \frac{5}{2}(r + \frac{24}{13}r) = 30$$

$$r = \frac{26}{17}$$

Thus the radii of the circles of Cases (i),(ii), and (iii), are $\frac{3}{2}$, $\frac{10}{9}$, and $\frac{26}{17}$, respectively.