USA Mathematical Talent Search

Solutions to Problem 1/1/16

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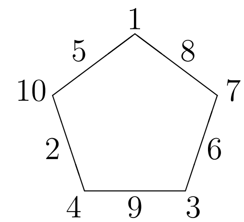
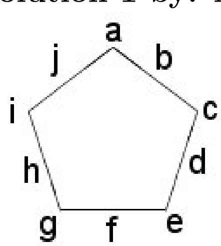
1/1/16. The numbers 1 through 10 can be arranged along the vertices and sides of a pentagon so that the sum of the three numbers along each side is the same. The diagram on the right shows an arrangement with sum 16. Find, with proof, the smallest possible value for a sum and give an example of an arrangement with that sum.



Credit This problem was invented by George Berzsenyi, inspired by Team Question 1 of the 2001 South East Asian Mathematics Olympiad.

Comments Some students solved this problem by pointing at that 10 must be on at least one side, so 10 + 1 + 2 = 13 is the smallest possible side sum. Such proofs go on to show that such a side sum is impossible, though this approach has the drawback of requiring many steps that must be discussed. Skipping any step led to a lack of rigor in many solutions. One student found a novel way to examine this approach by grouping the possible sums of 13 into those where the lowest number was 1, 2, or 3.

Solution 1 by: Aaron Pribadi (9/AZ)



The variables a through j are the values arranged around the pentagon.

Since the sums of the sides are the same:

$$a+b+c = c+d+e = e+f+g = g+h+i = i+j+a = \text{the sum of one side}$$

Thus,

$$\frac{(a+b+c)+(c+d+e)+(e+f+g)+(g+h+i)+(i+j+a)}{5} = \text{the sum of one side}$$

or it can be rearranged to

$$\frac{(a+b+c+d+e+f+g+h+i+j)+(a+c+e+g+i)}{5}$$

Since $a+b+c+d+e+f+g+h+i+j$ consists of the sum of the numbers one through 10, it equals $1+2+3+4+5+6+7+8+9+10$, or 55. Thus

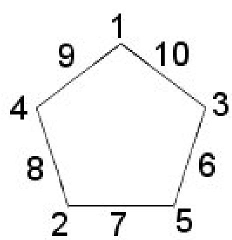
$$\frac{(55)+(a+c+g+e+i)}{5} = 11 + \frac{a+c+g+e+i}{5} = \text{the sum of one side}$$

The numbers are positive, so to minimize the expression above, $a + c + g + e + i$ must be minimized.
It is obvious that the values to minimize the sum $a + c + g + e + i$ are the numbers 1–5 out of the possible numbers of 1 through 10, since those are the 5 smallest numbers. So,

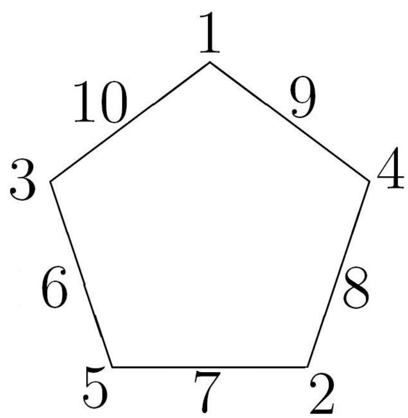
$$11 + \frac{1 + 2 + 3 + 4 + 5}{5} = 11 + 3 = 14 \text{ the minimum sum of one side}$$

14 is the smallest possible value for a sum.

Now that an optimal minimum value has been found, an example will show that the value can be attained. One such example is in the following diagram:



Solution 2 by: Nathan Pflueger (12/WA)
Define S for any arrangement of the numbers 1 through 10 on the edges and vertices of a pentagon to be the sum of the sums of the three numbers along each side of the pentagon. Notice that, in this summation, every number found on an edge will be added once (when the numbers on that edge are added), but each number on a vertex will be added twice (once for each edge connected to that vertex). Thus $S = \Sigma(\text{edge numbers}) + 2\Sigma(\text{vertex numbers})$. However, observe that $\Sigma(\text{edge numbers}) + \Sigma(\text{vertex numbers}) = \Sigma(\text{all numbers})$, and the sum of all numbers is the sum $1 + 2 + \cdots + 10 = 55$, thus $S = 55 + \Sigma(\text{vertex numbers})$. This can be minimized by making the vertex numbers 1 through 5, thus $S \geq 55 + 15 = 70$. If the sum of the numbers on each edge are equal, then each such sum is $S/5$, thus the least possible such sum is $70/5 = 14$. Indeed, this sum is possible by using the arrangement shown below.


1
10
9
3
4
6
8
5
7
2

2/1/16. For the equation

$$(3x^2 + y^2 - 4y - 17)^3 - (2x^2 + 2y^2 - 4y - 6)^3 = (x^2 - y^2 - 11)^3,$$

determine its solutions (x, y) where both x and y are integers. Prove that your answer lists all the integer solutions.

Credit The original version of the problem was invented by Dr. George Berzsenyi, the creator of the USAMTS, and was modified into its current form by Dr. Erin Schram of NSA.

Comments Nearly all correct solutions followed one of the three strategies outlined in the published solutions below. Solutions 1 and 2 by Tony Liu and Zachary Abel exhibit two methods using substitution and algebraic manipulation. Solution 3 by Meir Lakhovsky employs Fermat's Last Theorem.

Solution 1 by: Tony Liu (10/IL)
To simplify the algebra, let us denote

$$\begin{aligned} a &= 3x^2 + y^2 - 4y - 17 \\ b &= 2x^2 + 2y^2 - 4y - 6 \end{aligned}$$

so that

$$a - b = x^2 - y^2 - 11$$

It follows that the original equation is equivalent to

$$\begin{aligned} a^3 - b^3 &= (a-b)^3 \\ a^3 - b^3 &= a^3 - 3a^2b + 3ab^2 - b^3 \\ 0 &= -3a^2b + 3ab^2 \\ 0 &= 3ab(b-a) \end{aligned}$$

Now, we will have solutions if and only if $a = 0$, $b = 0$, or $a = b$.

Case 1: If $a = 0$, then

$$\begin{aligned} 3x^2 + y^2 - 4y - 17 &= 0 \\ 3x^2 + (y-2)^2 &= 21 \end{aligned}$$

Because the right hand side is divisible by 3, we conclude that $3|(y-2)^2$. Since squares are nonnegative, it follows that $(y-2)^2 = 0$ or 9. If $(y-2)^2 = 0$, then we have $3x^2 = 21$, which has no integral solutions. If $(y-2)^2 = 9$, then $y = -1$ or 5, and $3x^2 = 12$, so $x = \pm 2$. Our solutions (x, y) are thus

$$(-2,-1) \quad (-2,5) \quad (2,-1) \quad (2,5)$$

Case 2: If $b = 0$, then

$$\begin{aligned} 2x^2 + 2y^2 - 4y - 6 &= 0 \\ x^2 + y^2 - 2y - 3 &= 0 \\ x^2 + (y-1)^2 &= 4 \end{aligned}$$

Since the only squares less than or equal to 4 are $0, 1$, and 4, we must have one term equal to 0 and the other equal to 4. If $x^2 = 0$, then $(y-1)^2 = 4$, and $y = -1$ or 3. If $x^2 = 4$, then $x = \pm 2$ and $(y-1)^2 = 0$, so $y = 1$. Thus our solutions for this case are

$$(0,-1) \quad (0,3) \quad (-2,1) \quad (2,1)$$

Case 3: If $a = b$, then

$$\begin{aligned} 3x^2 + y^2 - 4y - 17 &= 2x^2 + 2y^2 - 4y - 6 \\ x^2 - y^2 &= 11 \\ (x+y)(x-y) &= 11 \end{aligned}$$

From this equation, we note that 11 is prime to conclude that $x + y = \pm 1$ or ± 11, and $x - y = \pm 11$ or ± 1. Solving these four systems of equations, we obtain the solutions

$$(-6,-5) \quad (-6,5) \quad (6,-5) \quad (6,5)$$

Our final solution list is thus

$$\begin{array}{cccc} (-2,-1) & (-2,5) & (2,-1) & (2,5) \\ (0,-1) & (0,3) & (-2,1) & (2,1) \\ (-6,-5) & (-6,5) & (6,-5) & (6,5) \end{array}$$

Since we have considered all of the cases, these are indeed all the solutions. ■

Solution 2 by: Zachary Abel (11/TX)
If we let $a = x^2 - y^2 - 11$ and $b = 2x^2 + 2y^2 - 4y - 6$, then we (conveniently) have $a + b = 3x^2 + y^2 - 4y - 17$. So the given equation becomes

$$0 = (a + b)^3 - a^3 - b^3 = 3ab(a + b)$$

Thus, either $a = 0$, $b = 0$, or $a + b = 0$. We do these cases separately.

Case 1: $\mathbf{a = 0}$.
We have $(x - y)(x + y) = 11$. But since 11 decomposes into the product of 2 integers in only four ways, we obtain the following systems of equations:

$$\left\{\begin{array}{rcl} x-y & = & 1 \\ x+y & = & 11 \end{array}\right. ; \quad \left\{\begin{array}{rcl} x-y & = & 11 \\ x+y & = & 1 \end{array}\right. ; \quad \left\{\begin{array}{rcl} x-y & = & -1 \\ x+y & = & -11 \end{array}\right. ; \quad \left\{\begin{array}{rcl} x-y & = & -11 \\ x+y & = & -1 \end{array}\right.$$

These systems give four solutions: $(x, y) = (6, 5)$, $(6, -5)$, $(-6, 5)$, and $(-6, -5)$.

Case 2: $\mathbf{b = 0}$.
The equation $2x^2 + 2y^2 - 4y - 6 = 0$ is equivalent to $x^2 + (y - 1)^2 = 4$. There are only a few cases.

- If $|x| = 0$, then $(y - 1)^2 = 4$, giving the solutions $(x, y) = (0, 3)$ and $(0, -1)$.
- If $|x| = 1$, then $(y - 1)^2 = 3$, which is not solvable in integers.
- If $|x| = 2$, then $(y - 1)^2 = 0$, giving the solutions $(x, y) = (2, 1)$ and $(-2, 1)$.
- If $|x| > 2$, then $0 \le (y - 1)^2 = 4 - x^2 < 0$, which is impossible.

Case 3: $\mathbf{a + b = 0}$.
We have $3x^2 + y^2 - 4y - 17 = 0$, i.e. $3x^2 + (y - 2)^2 = 21$. There are again 4 cases:

- If $|x| = 0$, then $(y - 2)^2 = 21$, which is impossible in integers.
- If $|x| = 1$, then $(y - 2)^2 = 18$, impossible in integers.
- If $|x| = 2$, then $(y - 2)^2 = 9$, leading to $(x, y) = (2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$.
- If $|x| \ge 3$, then $(y - 2)^2 = 21 - 3x^2 \le 21 - 3 \cdot 3^2 < 0$, which is impossible since squares are non-negative.

So there are 12 solutions: $(x, y) = (6, 5)$, $(6, -5)$, $(-6, 5)$, $(-6, -5)$, $(0, 3)$, $(0, -1)$, $(2, 1)$, $(-2, 1)$, $(2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$.

Solution 3 by: Meir Lakhovsky (9/WA)
Fermat's Last Theorem states that $a^n + b^n = c^n$ has integer solutions for $n \geq 3$ if and only if a, b, and/or $c = 0$. We let $(3x^2 + y^2 - 4y - 17) = c$; $(2x^2 + 2y^2 - 4y - 6) = b$; and $(x^2 - y^2 - 11) = a$, and note that our equation has the form $a^3 + b^3 = c^3$. Thus, either $[a = 0, b = c \neq 0]$ or $[b = 0, a = c \neq 0]$ or $[\ c = 0, a = -b \neq 0\ ]$ or $[\ a = b = c = 0\ ]$.

Case A: $a = 0, b = c \neq 0$
$x^2 - y^2 - 11 = 0 \Rightarrow (x - y)(x + y) = 11$. Since 11 is prime, its only factorizations are $11 * 1$ and $(-11) * (-1)$; therefore $(x-y)$ and $(x+y)$ equal either: $(1, 11)$ or $(11, 1)$ or $(-1, -11)$ or $(-11, -1)$ respectively. Solving each case individually, we get, $(6, 5)$, $(6, -5)$, $(-6, -5)$, and $(-6, 5)$ respectively. Furthermore, in all the cases $b = c \neq 0$. But, we notice, that whenever $a = 0$, we have $b \neq 0$, thus the case $a = b = c = 0$ is impossible.

Case B: $b = 0, a = c \neq 0$
$2x^2 + 2y^2 - 4y - 6 = 0 \Rightarrow x^2 + y^2 - 2y - 3 = 0 \Rightarrow x^2 + (y - 1)^2 = 4$. Since no two non-zero squares add up to 4, either x^2 or $(y-1)^2$ equal 0, while the other equals 4. Thus, the integral solutions in this case are $(2, 1)$, $(-2, 1)$, $(0, 3)$, and $(0, -1)$. All of these solutions give us $a = c \neq 0$, thus, all of them are valid.

Case C: $c = 0, a = -b \neq 0$
$3x^2 + y^2 - 4y - 17 \Rightarrow 3(x^2) + (y - 2)^2 = 21$. Through a little trial and error (plugging 0,1, 4, and for x^2), we see that the only possible value of x^2 which leaves y integral is 4. This yields the solutions $(2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$. In all of these cases $a = -b \neq 0$, thus, they are all valid.

There are no other integral solutions because we covered every case which follows from Fermat's Last Theorem. In conclusion, there are 12 solutions: $(6, 5)$, $(6, -5)$, $(-6, -5)$, $(-6, 5)$, $(2, 1)$, $(-2, 1)$, $(0, 3)$, $(0, -1)$, $(2, 5)$, $(2, -1)$, $(-2, 5)$, and $(-2, -1)$.

3/1/16. Given that $5r + 4s + 3t + 6u = 100$, where $r \geq s \geq t \geq u \geq 0$ are real numbers, find, with proof, the maximum and minimum possible values of $r + s + t + u$.

Credit This problem was inspired by a similar problem posed 35 years ago in the first round of Hungary's Dániel Arany Mathematical Competition for students of advanced standing.

Comments Three elegant algebraic solutions are presented below. Some students also solved this problem by considering the region of 4–dimensional space described by the inequalities $r \geq s \geq t \geq u \geq 0$. The minimum and maximum of $r+s+t+u$ must located at the 'corners' of this space. Thus, we must test $(x, 0, 0, 0)$; $(x, x, 0, 0)$; $(x, x, x, 0)$; and (x, x, x, x) by finding the value of x in each case which satisfies the given $5r + 4s + 3t + 6u = 100$ and evaluating $r + s + t + u$ at the resulting points.

Solution 1 by: Yakov Berchenko-Kogan (10/NC)
Let:

$$\begin{array}{ccccccccc} u & + & a & & & & & = & t \\ u & + & a & + & b & & & = & s \\ u & + & a & + & b & + & c & = & r \end{array}$$

Since $r \geq s \geq t \geq u \geq 0$, we know $a, b, c \in \mathbb{R}_0^+$. Note that:

$$r + s + t + u = 4u + 3a + 2b + c$$

Substituting:

$$5r + 4s + 3t + 6u = 100$$

$$5(u + a + b + c) + 4(u + a + b) + 3(u + a) + 6u = 100$$

$$18u + 12a + 9b + 5c = 100$$

$$(2u + b + c) + 4(r + s + t + u) = 100$$

Clearly, in order to maximize $r+s+t+u$ we must minimize $2u+b+c$. Since all values are positive, this can easily be done by setting $u = b = c = 0$. Now, we can find what exactly the maximum value is:

$$4(r + s + t + u) = 100$$

$$r + s + t + u = 25$$

Thus 25 is the maximum value of $r + s + t + u$, achieved when $r = s = t = \frac{25}{3}$ and $u = 0$.

Now we must find the minimum value:

$$18u + 12a + 9b + 5c = 100$$

$$5(r + s + t + u) - (2u + 3a + b) = 100$$

Similarly to before, in order to minimize $r+s+t+u$ we must minimize $2u+3a+b$, and this is easily done by setting $u=a=b=0$. Again, we can easily find what exactly the minimum value is:

$$5(r+s+t+u)=100$$
$$r+s+t+u=20$$

Thus the minimum value of $r+s+t+u$ is 20, achieved when $r=20$ and $s=t=u=0$.

So, in summary, $20 \le r+s+t+u \le 25$.

Solution 2 by: Zachary Abel (11/TX)

Define $S=r+s+t+u$. Since $r \ge s \ge t \ge u \ge 0$, the numbers $r-s,\ s-t,\ t-u$, and u are non-negative. To find the lower bound, we calculate as follows:

$$\begin{aligned} S &= r+s+t+u \\ &= (r-s)+2(s-t)+3(t-u)+4u \\ &\ge (r-s)+\tfrac{9}{5}(s-t)+\tfrac{12}{5}(t-u)+\tfrac{18}{5}u \\ &= \tfrac{1}{5}\big(5r+4s+3t+6u\big) \\ &= \tfrac{1}{5}\big(100\big) \\ &= 20. \end{aligned}$$

The minimum of 20 can be achieved when $(r,s,t,u)=(20,0,0,0)$. We similarly find the upper bound:

$$\begin{aligned} S &= r+s+t+u \\ &= (r-s)+2(s-t)+3(t-u)+4u \\ &\le \tfrac{5}{4}(r-s)+\tfrac{9}{4}(s-t)+3(t-u)+\tfrac{9}{2}u \\ &= \tfrac{1}{4}\big(5r+4s+3t+6u\big) \\ &= \tfrac{1}{4}\big(100\big) \\ &= 25. \end{aligned}$$

This maximum is attained when $(r,s,t,u)=(\frac{25}{3},\frac{25}{3},\frac{25}{3},0)$. Thus, the minimum and maximum values of S are 20 and 25 respectively.

Solution 3 by: Feiqi Jiang (9/MA)
Since $r \geq t$, we have $r - t \geq 0$. Also, $u \geq 0$ implies $2u \geq 0$. Adding this to $r - t \geq 0$ gives $r - t + 2u \geq 0$

Note that

$$4(r + s + t + u) + (r - t + 2u) = 5r + 4s + 3t + 6u = 100.$$

Therefore,

$$\begin{aligned} 100 - 4(r + s + t + u) = (r - t + 2u) &\geq 0 \\ 100 - 4(r + s + t + u) &\geq 0 \\ 100 &\geq 4(r + s + t + u) \\ 25 &\geq r + s + t + u \end{aligned}$$

Hence the maximum value of $r + s + t + u$ is 25.

We take a similar approach for the minimum: $s \geq u$ implies $s - u \geq 0$. Adding this to $2t \geq 0$ gives $s - u + 2t \geq 0$.

Note that

$$5(r + s + t + u) - (s - u + 2t) = 5r + 4s + 3t + 6u = 100.$$

Therefore

$$\begin{aligned} 5(r + s + t + u) - 100 = s - u + 2t &\geq 0 \\ 5(r + s + t + u) - 100 &\geq 0 \\ 5(r + s + t + u) &\geq 100 \\ r + s + t + u &\geq 20 \end{aligned}$$

Thus the minimum value of $r + s + t + u$ is 20.

4/1/16. The interior angles of a convex polygon form an arithmetic progression with a common difference of 4°. Determine the number of sides of the polygon if its largest interior angle is 172°.

Credit This problem was modeled after Problem 106 on page 75 of Volume 3, Number 1 (Spring 1981) of AGATE, which was edited in Alberta, Canada, by Professor Andy Liu, a long-time friend of the USAMTS program. Dr. Liu referenced the *Second Book of Mathematical Bafflers*, edited by A. F. Dunn and published by Dover Publications in 1983.

Comments The most straightforward solution is as in Solution 1. A slightly more elegant solution, using exterior angles instead of interior angles, is shown in Solution 2. Note that both method leads to the same quadratic equation. It is also possible to use a "guess and check" method on this problem.

Solution 1 by: Yakov Berchenko-Kogan (10/NC)

Let n be the number of sides of the polygon. The first step to solving this problem is to determine the sum of the interior angles determined by the arithmetic progression in terms of n. It is fairly clear that the progression is $172, 168, 164, ..., 172 - 4(n-1)$. Thus we can find its sum to be:

$$\begin{aligned} & 172n - 4\frac{n(n-1)}{2} \\ = \; & 172n - 2n(n-1) \\ = \; & 174n - 2n^2. \end{aligned}$$

We also know the formula for the sum of the interior angles of an n-sided polygon: $180(n-2) = 180n - 360$. Thus we can equate these two and solve for n:

$$180n - 360 = 174n - 2n^2$$

$$2n^2 + 6n - 360 = 0$$

$$n^2 + 3n - 180 = 0$$

$$(n+15)(n-12) = 0$$

Obviously $n = -15$ is an extraneous solution, and so we know that $n = 12$, and thus the polygon is a dodecagon.

Solution 2 by: Benjamin Lee (9/MD)

The sum of the exterior angles of the polygon is always 360°. So,

$$\text{Sum}_{exterior\ angles} = 8 + (8+4(1)) + (8+4(2)) + \cdots + (8+4(n-2)) + (8+4(n-1)) = 360,$$

where n is the number of sides of the polygon.

The equation can be simplified to

$$
\begin{aligned}
8n + 4(1 + 2 + \cdots + (n-2) + (n-1)) &= 360 \\
2n + \frac{(n-1)(n)}{2} &= 90 \\
4n + (n-1)(n) &= 180 \\
n^2 + 3n - 180 &= 0 \\
(n-12)(n+15) &= 0
\end{aligned}
$$

Therefore $n = 12$ or $n = -15$. Since n must be a positive integer, $n = 12$ and we are done.

USA Mathematical Talent Search

Solutions to Problem 5/1/16

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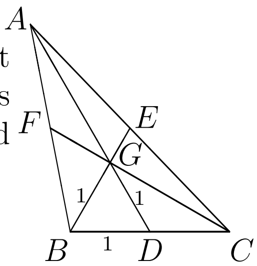


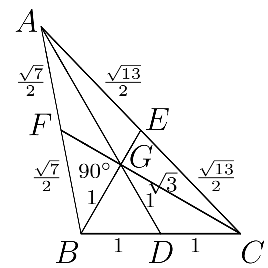
5/1/16. Point G is where the medians of the triangle ABC intersect and point D is the midpoint of side $\overline{BC}$. The triangle BDG is equilateral with side length 1. Determine the lengths, AB, BC, and CA, of the sides of triangle ABC.

Credit The problem was proposed by Professor Gregory Galperin, a member of the Committee in charge of the USA Mathematical Olympiad. He has suggested many excellent problems to the USAMTS over the years.

Comments Our participants found numerous solutions to this problem. Below are 4 of the most common approaches. Other solutions involved complex numbers, clever rotations or line extensions, and complicated analytic geometry. Our first solution below, from Timothy Zhu, exhibits the basic geometric approach. The second solution, from David Benjamin, employs the law of cosines, while Lawrence Chan gives us a third solution using analytic geometry. Solution 4 from Ameya Velingker uses Stewart's Theorem to prove a formula relating median lengths to the side lengths of a triangle, and Solution 5 from Joshua Horowitz uses vectors and analytic geometry.

Solution 1 by: Timothy Zhu (12/NH)
Lemma 1: $\overline{BE} \perp \overline{CF}$
Proof: Since D is the midpoint of $\overline{BC}$, $\overline{DC}$ has side length 1. Since $\triangle BGD$ is equilateral, its angles are all $60°$. Therefore, $\angle GDC = 180° - 60° = 120°$. Since $\overline{DG} = \overline{DC}$, $\triangle GDC$ is isosceles. Thus, $\angle DGC$ and $\angle DCG$ are both $\frac{180° - \angle GDC}{2} = 30°$, so $\angle BGC = \angle BGD + \angle DGC = 90°$. Therefore, $\overline{BE} \perp \overline{CF}$.

——end Lemma 1——



Since $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are medians, D, E, and F are midpoints. Therefore,

$$\overline{AB} = 2\overline{FB}$$

$$\overline{BC} = 2$$

$$\overline{CA} = 2\overline{EC}$$

Since $\overline{BE} \perp \overline{CF}$ (Lemma 1), $\overline{FB}$, $\overline{EC}$, and $\overline{GC}$ can be calculated using the Pythagorean Theorem.

$$\overline{AB} = 2\overline{FB} = 2\sqrt{\overline{BG}^2 + \overline{GF}^2}$$

$$\overline{BC} = 2$$

$$\overline{CA} = 2\overline{EC} = 2\sqrt{\overline{GC}^2 + \overline{GE}^2}$$

$$\overline{GC} = \sqrt{\overline{BC}^2 - \overline{BG}^2} = \sqrt{2^2 - 1^2} = \sqrt{3}$$

It is well known that the centroid, G, divides the medians in 2:1 ratios. Thus,

$$\overline{GF} = \frac{\overline{GC}}{2} \text{ and } \overline{GE} = \frac{\overline{BG}}{2}$$

So,

$$\overline{AB} = 2\overline{FB} = 2\sqrt{\overline{BG}^2 + \overline{GF}^2} = 2\sqrt{\overline{BG}^2 + \left(\frac{\overline{GC}}{2}\right)^2} = 2\sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \sqrt{7}$$

$$\overline{BC} = 2$$

$$\overline{CA} = 2\overline{EC} = 2\sqrt{\overline{GC}^2 + \overline{GE}^2} = 2\sqrt{\overline{GC}^2 + \left(\frac{\overline{BG}}{2}\right)^2} = 2\sqrt{\sqrt{3}^2 + \left(\frac{1}{2}\right)^2} = \sqrt{13}$$

Solution 2 by: David Benjamin (8/IN)

Side BC

D is the midpoint of $\overline{BC}$, so $DC = 1$, and $BC = 2$.

Side AB

G is the centroid of the triangle, so $\frac{AG}{GD} = 2$, so $AG = 2$, so $AD = 3$.

$\triangle BDG$ is an equilateral triangle, so $\angle GDB = 60°$.

So, by the Law of Cosines,

$$\begin{aligned}
(AB)^2 &= (BD)^2 + (AD)^2 - 2(AD)(BD)\cos\angle GDB \\
&= 1^2 + 3^2 - 2 \times 3 \times 1 \times \cos 60° \\
&= 1 + 9 - 6 \times \frac{1}{2} \\
&= 10 - 3 \\
&= 7 \\
AB &= \sqrt{7}
\end{aligned}$$

Side AC

$\angle GDC$ and $\angle GDB$ are supplementary, so $\angle GDC = 120°$.

So, by the Law of Cosines,

$$\begin{aligned}
(AC)^2 &= (DC)^2 + (AD)^2 - 2(DC)(AD)\cos\angle GDC \\
&= 1^2 + 3^2 - 2 \times 1 \times 3 \times \cos 120° \\
&= 1 + 9 - 6 \times \left(-\frac{1}{2}\right) \\
&= 10 + 3 \\
&= 13 \\
AC &= \sqrt{13}
\end{aligned}$$

Solution 3 by: Lawrence Chan (11/IL)

We will solve this problem using a coordinate plane with B as the origin and an x-axis running parallel to $\overline{BC}$. $\overline{BD} = \overline{DC}$ because $\overline{AD}$ is a median, so the coordinates of C are twice those of D. Since $D = (1, 0)$, $C = (2, 0)$. $\angle GBD = 60^\circ$ and $\overline{BG} = 1$ because $\triangle BGD$ is equilateral with side length 1, so using trigonometry we get

$$G = (\cos 60^\circ, \sin 60^\circ) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$$

G is the centroid, since it is the intersection of the medians. Centroids have the property that their coordinates are the averages of the coordinates of the vertices. Let $A = (a_1, a_2)$, $B = (0, 0)$, and $C = (2, 0)$ be our three vertices. Since the centroid $G = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ has coordinates that are the averages of the respective coordinates in the vertices, we can set up two equations as follows:

$$\begin{aligned} \frac{1}{2} &= \left(\frac{a_1 + 0 + 2}{3}\right) \\ \frac{\sqrt{3}}{2} &= \left(\frac{a_2 + 0 + 0}{3}\right) \end{aligned}$$

From these we can deduce that $A = \left(-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right)$

Applying the distance formula to the three sides of the triangle gives us

$$\boxed{\overline{AB} = \sqrt{7},\ \overline{BC} = 2,\ \overline{CA} = \sqrt{13}}$$

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Solution 4 by: Ameya Velingker (11/PA)

Let $m_A = AD$, $m_B = BE$, $m_C = CF$ be the lengths of the medians from A, B, and C, respectively. Also, for convenience, we let $a = BC$, $b = CA$, and $c = AB$ be the sides of the triangle. By Stewart's Theorem,

$$\begin{aligned} a\left({m_A}^2 + BD \cdot CD\right) &= b^2 \cdot BD + c^2 \cdot CD \\ a\left({m_A}^2 + \frac{a^2}{4}\right) &= b^2 \cdot \frac{a}{2} + c^2 \cdot \frac{a}{2} \\ m_A &= \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2} \qquad (1) \end{aligned}$$

Clearly, $a = 2BD = 2$ and $m_A = 3DG = 3$ since the centroid divides each median in a $2:1$ ratio. Substituting these expressions in to (1) and simplyfying, we obtain $b^2 + c^2 = 20$. Now, we note the equation

$$m_B = \frac{1}{2}\sqrt{2c^2 + 2a^2 - b^2} \qquad (2)$$

which can be derived in the same way we derived (1). Observe that $m_B = \frac{3}{2}BG = \frac{3}{2}$. Substituting this expression along with $a = 2$ into (2) and rearranging, we find that $2c^2 - b^2 = 1$. Solving this equation simultaneously with $b^2 + c^2 = 20$, we get $b = \sqrt{13}$ and $c = \sqrt{7}$. Thus, the side lengths of the triangle are 2, $\sqrt{13}$, and $\sqrt{7}$.

Solution 5 by: Joshua Horowitz (10/CT)
We apply coordinates to our diagram, so that $B = (0,0)$, $D = (1,0)$, and $C = (2,0)$. BDG is equilateral, so $G = (\frac{1}{2}, \frac{\sqrt{3}}{2})$. But also, since G is the centroid of ABC, $\vec{G} = (\vec{A}+\vec{B}+\vec{C})/3$ (using vector notation). We can use this to solve for A:

$$\begin{aligned}
\vec{G} &= \frac{\vec{A}+\vec{B}+\vec{C}}{3} \\
\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) &= \frac{\vec{A}+(0,0)+(2,0)}{3} \\
\left(\frac{3}{2}, \frac{3\sqrt{3}}{2}\right) &= \vec{A}+(2,0) \\
\left(-\frac{1}{2}, \frac{3\sqrt{3}}{2}\right) &= \vec{A}.
\end{aligned}$$

Determining side lengths is now just an application of the distance formula:

$$\begin{aligned}
AB &= \sqrt{\left(-\frac{1}{2}-0\right)^2 + \left(\frac{3\sqrt{3}}{2}-0\right)^2} = \sqrt{7} \\
BC &= \sqrt{(0-2)^2+(0-0)^2} = 2 \\
CA &= \sqrt{\left(2-\left(-\frac{1}{2}\right)\right)^2 + \left(0-\frac{3\sqrt{3}}{2}\right)^2} = \sqrt{13}.
\end{aligned}$$