USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS

Round 2 - Year 14 - Academic Year 2002-2003

1/2/14. Each member of the sequence 112002, 11210, 1121, 117, 46, 34, … is obtained by adding five times the rightmost digit to the number formed by omitting that digit. Determine the billionth (10^9th) member of this sequence.

Comments: This problem was based on a problem from issue 3/2001 of Matlap, Transylvania's excellent Hungarian language mathematics journal for students at the middle school and high school levels.

Solution for 1/2/14 by Aleksandr Arkhipov (9/NJ):

This sequence generates a new member by taking a function, $f(x)$, of the preceding member x, so that $f(n\text{th member})$ is the $(n+1)$ st member. The function $f(x)$ can be simplified if we express the integer x as $10k + y$, where $y = x \bmod 10$; *i.e.,* y is the ones digit of x. Then we can say that $f(10k + y) = k + 5y$, since the function removes the last digit y, leaving behind the integer part of $x/10$, which is k, and adds five times that last digit y.

Next, the value of any member of the sequence depends only on the value of the preceding member. Therefore, as soon as any member appears that is equal to any previous member in the sequence, the rest of the series will repeat again and again generating the same value every Nth time, where N is the distance between the original two identical members. This leads to three possibilities:

1) The sequence settles on one number x for which $f(x) = x$.
2) The squence enters a cycle in which a block of N numbers repeats over and over again.
3) No member ever repeats.

In option 3 the sequence increases boundlessly toward infinity or bounces around without repeating in an infinite range, since its range has to hold infinitely many values. However, whenever any member of the sequence is under 49, all members thereafter will be under 49. Therefore, the sequence will stay under 49 after the fifth member 46. So option 3 is ruled out.

Forty-nine is the maximum value of $f(x)$ for $x \le 49$, because if $k \le 4$ in the $10k + y$ form of x, then $f(10k + y) = k + 5y \le 4 + 5(9) = 49$.

Next, I wrote a simple computer program to output the sequence. I found the sequence started with the nonrepeating members 112002, 11210, 1121, and 117, and after that a block of 42 numbers (46, 34, 23, 17, 36, 33, 18, 41, 9, 45, 29, 47, 39, 48, 44, 24, 22, 12, 11, 6, 30, 3, 15, 26, 32, 13, 16, 31, 8, 40, 4, 20, 2, 10, 1, 5, 25, 27, 37, 38, 43, 19) repeats. This means that the 5th, 47th, and 89th members were all 46, continuing to all positions that equal 5 modulo 42 (that means the position n can be expressed as $42i + 5$ for some integer i). By the same logic, since 10^9 reduced modulo 42 equals 34, ($10^9 = 238095238 \times 42 + 34$), the billionth member is the same as as the 34th member, which is 40.

2/2/14. The integer 72 is the first of three consecutive integers 72, 73, and 74, that can each be expressed as the sum of the squares of two positive integers. The integers 72, 288, and 800 are the first three members of an infinite increasing sequence of integers with the above property. Find a function that generates the sequence and give the next three members.

Comment: This problem is based on a similar problem that appeared in the 1991 Indiana College Mathematics Competition, initiated by Professor Paul Mielke.

Solution 1 for 2/2/14 by George Lin (11/NJ):

We want a function $f(n)$ such that $f(1) = 72$, $f(2) = 288$, $f(3) = 800$, and $f(n)$, $f(n)+1$, and $f(n)+2$ can all be expressed as the sum of the squares of two positive integers for all positive integers n. First, let us write out the known three sets of consecutive integers as the sum of the squares of two positive integers:

$f(1)$	$f(2)$	$f(3)$
$72 = 6^2 + 6^2$	$288 = 12^2 + 12^2$	$800 = 20^2 + 20^2$
$73 = 3^2 + 8^2$	$289 = 8^2 + 15^2$	$801 = 15^2 + 24^2$
$74 = 5^2 + 7^2$	$290 = 11^2 + 13^2$	$802 = 19^2 + 21^2$

Notice that all of the first of the three consecutive integers, 72, 288, and 800, are of the form $a^2 + a^2$. Thus, assume $f(n) = a_n^2 + a_n^2 = 2a_n^2$.

Notice the all the last of the three consecutive integers, 74, 290, and 802, are of the form $(a-1)^2 + (a+1)^2$. If we expand this out, we can see this clearly works for any $f(n) = 2a_n^2$:

$$f(n) + 2 = (a_n - 1)^2 + (a_n + 1)^2 = (a_n^2 - 2a_n + 1) + (a_n^2 + 2a_n + 1) = 2a_n^2 + 2.$$

Now, we need only deal with the second of the three consecutive integers. We have $a_1 = 6$, $a_2 = 12$, and $a_3 = 20$, so

$$f(1) + 1 = (a_1 - 3)^2 + (a_1 + 2)^2$$

$$f(2) + 1 = (a_2 - 4)^2 + (a_2 + 3)^2$$

$$f(3) + 1 = (a_3 - 5)^2 + (a_3 + 4)^2$$

Based on the first three members of the sequence $f(n)+1$, we extrapolate the trend and assume that $f(n) + 1 = (a_n - (n+2))^2 + (a_n + (n+1))^2$. This gives

$$\begin{aligned} f(n) + 1 &= (a_n - (n+2))^2 + (a_n + (n+1))^2 \\ &= a_n^2 - 2a_n(n+2) + (n+2)^2 + a_n^2 + 2a_n(n+1) + (n+1)^2 \\ &= 2a_n^2 - 2a_n + 2n^2 + 6n + 5 \end{aligned}$$

Since $f(n)+1 = 2a_n^2+1$, we get the equation

$$2a_n^2+1 = 2a_n^2 - 2a_n + 2n^2 + 6n + 5$$
$$2a_n = 2n^2 + 6n + 4$$
$$a_n = n^2 + 3n + 2$$

Therefore, we can let $a_n = n^2 + 3n + 2$ and $f(n) = 2a_n^2$, which gives

$$f(n) = a_n^2 + a_n^2$$
$$f(n)+1 = (a_n - (n+2))^2 + (a_n + (n+1))^2$$
$$f(n)+2 = (a_n - 1)^2 + (a_n + 1)^2,$$

and $f(1) = 72$, $f(2) = 288$, and $f(3) = 800$. Letting $n = 4, 5, 6$, we find the next three terms in the sequence: 1800, 3528, 6272.

Solution 2 for 2/2/14 by Sherry Gong (8/PR):

Answer: The function is generated by $2(n+1)^2(n+2)^2$. The next three terms are 1800, 3528, and 6272.

Proof:

$$\begin{aligned}
2(n+1)^2(n+2)^2 &= ((n+1)(n+2))^2 + ((n+1)(n+2))^2 \\
&= 2n^4 + 12n^3 + 26n^2 + 24n + 8 \\
2(n+1)^2(n+2)^2 + 1 &= (n(n+2))^2 + ((n+1)(n+3))^2 \\
&= 2n^4 + 12n^3 + 26n^2 + 24n + 9 \\
2(n+1)^2(n+2)^2 + 2 &= ((n+1)(n+2) - 1)^2 + ((n+1)(n+2) + 1)^2 \\
&= 2n^4 + 12n^3 + 26n^2 + 24n + 10
\end{aligned}$$

3/2/14. An integer lattice point in the Cartesian plane is a point (x, y) where x and y are both integers. Suppose nine integer lattice points are chosen such that no three of them lie on the same line. Out of all 36 possible line segments between pairs of those nine points, some line segments may contain integer lattice points besides the original nine points. What is the minimum number of line segments that must contain an integer lattice point besides the original nine points? Prove your answer.

Comment: This problem was modified by Gene Berg and George Berzsenyi from a problem suggested by Professor Tom Hoeholdt of the Technical University of Denmark.

Solution 1 for 3/2/14 by Irena Wang (11/IL):

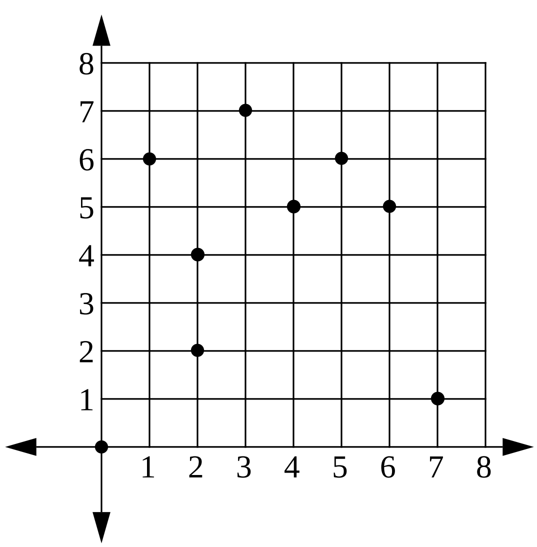
Answer: 6

If two points (a, b) and (c, d) have lattice points on the segment between them, then $a - c$ and $b - d$ must both be divisible by some common factor q with $q > 1$. So $a \equiv c \pmod{q}$ and $b \equiv d \pmod{q}$, which we can write as $(a, b) \equiv (c, d) \pmod{q}$. For example, $(10, 2) \equiv (3, 9) \pmod{7}$.

For modulo *n* where *n* is 3 or higher, it is possible for nine points to have no congruent pairs modulo *n*, since there are n^2 different possible types of points modulo *n*.

For modulo 2, however, there are only $2^2 = 4$ possiblities: (even, even), (even, odd), (odd, even), and (odd, odd), so there will be congruent pairs of points. For the least number of such pairs, we can make 3 points of one type, such as (even, even), and 2 points in each of the other types. Since the segments determined by congruent pairs of points contain lattice points, this will give us at least $\binom{3}{2} + \binom{2}{2} + \binom{2}{2} + \binom{2}{2} = 6$ segments containing lattice points.

Now that a lower bound has been established, we need a real example of nine points that determine only six segments containing other lattice points to complete the proof. One such example is $(0, 0)$, $(1, 6)$, $(2, 2)$, $(2, 4)$, $(3, 7)$, $(4, 5)$, $(5, 6)$, $(6, 5)$, and $(7, 1)$. Note that there are no two points congruent in any modulus beyond 2.



Solution 2 for 3/2/14 by Jonathan Sasmor (12/NY):

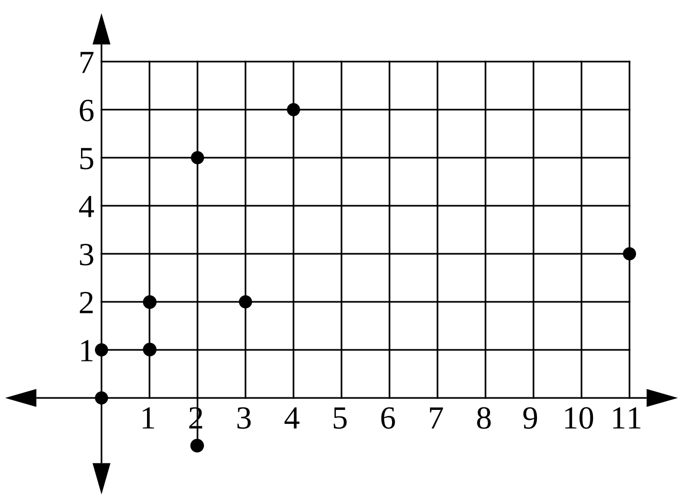
Two integers have the same parity if they are both even or both odd. Consider two lattice points to have the same parity if their *x*-coordinates have the same parity and their *y*-coordinates have the same parity. If two lattice points are of the same parity, the differences in their *x*- and *y*-coordinates will be even, and thus, by the midpoint formula, the midpoint of the line segment connecting them will be a lattice point. We seek to minimize the number of line segments connecting lattice points of the same parity.

Regardless of how we choose our nine lattice points, there will be at least six line segments connecting lattice points of the same parity.

- If there is any group of four or more lattice points of the same parity, there will be at least $\binom{4}{2} = 6$ line segments connecting these points.
- If there are two groups of three lattice points of the same parity, there must be at least $2\binom{3}{2} = 6$ line segments connecting the points in each group.
- Since there are only four lattice point parities: (even, even), (odd, odd), (even, odd), (odd, even)—and nine lattice points, then from the pigeonhole principle, there is at least one group of three points of the same parity. If there is only one such group, the other three parities must have two points each. The number of line segments with midpoint lattice points would be $\binom{3}{2} + 3\binom{2}{2} = 6$.

Thus, in all possible cases, there must be at least six line segments with lattice point midpoints.

For the set of lattice points $\{ (0, 0), (0, 1), (1, 1), (1, 2), (2, -1), (2, 5), (3, 2), (4, 6), (11, 3) \}$, no three points are collinear and exactly 6 of the 36 connecting line segments contain a lattice point besides the original nine points. Therefore the minimum possible number of such line segments is 6.



4/2/14. Let $f(n)$ be the number of ones that occur in the decimal representations of all the numbers from 1 to n. For example, this gives $f(8) = 1$, $f(9) = 1$, $f(10) = 2$, $f(11) = 4$, and $f(12) = 5$. Determine the value of $f(10^{100})$.

Comment: Professor Roger Pinkham of Stevens Institute of Technology suggested the original problem that used this function, and Dr. Michael Zieve of IDA/CCR simplified the suggestion into a problem appropriate for this competition.

Solution 1 for 4/2/14 by Daniel McLaury (12/OK):

It is helpful to think of numbers between 0 and $10^{100} - 1$ as strings of exactly 100 digits. For example, 1417 would be represented as 0000…00001417. Then simple combinatorics show a pretty simple solution.

Note that $10^{100} - 1$ is 9999…9999, all its digits are 9. Observe that exactly one tenth of the numbers between 0 and $10^{100} - 1$ will have a 1 in the one's place; similarly, one tenth will have a 1 in the ten's place, etc., and the pattern continues all the way up to the 10^{99}'s place. (We are free to add 0 to the list of numbers that we count 1's in, because it has no 1's in it and therefore does not affect the total.)

Since there are 100 places in each number (the 10^0's place, the 10^1's place, the 10^2's place, up to the 10^{99}'s place) and in each place $100^{10}/10 = 10^{99}$ many 1's will show up, there are $100 \times 10^{99} = 10^{101}$ many 1's in the numbers between 0 and $10^{100} - 1$. Then all that is left to count is the single 1 in the googol's place in 10^{100} itself. Thus, there are $10^{101} + 1$ many 1's in all. So $f(10^{100}) = 10^{101} + 1$.

In general, $a10^{a-1} + 1$ many 1's are used in writing all the numbers from 1 to 10^a.

Solution 2 for 4/2/14 by Daniel Walton (12/WA):

Using combinatorics, I found $f(10^1)$ through $f(10^{10})$. Here is an example of my method for $f(10^3)$, *i.e.,* $f(1000)$:

> A number between 1 and 1000 contains zero ones, one one, two ones, or three ones. Consider all numbers between 1 and 1000 that contain exactly *n* ones. The number of ones found in all such numbers is
>
> $n \times (\text{The number of ways to position } n \text{ ones}) \times 9^{(3-n)}$,
>
> except when n is one, in which case 1000 itself gives an additional one.
>
> • For numbers with only one one (besides 1000), that is $1 \times \binom{3}{1} \times 9^2 = 243$.
>
> • For numbers with exactly two ones, that is $2 \times \binom{3}{2} \times 9^1 = 54$.
>
> • For numbers with exactly three ones, that is $3 \times \binom{3}{3} \times 9^0 = 3$.

- There is another one in the number 1000, that is 1.

$243 + 54 + 3 + 1 = 301$.

I observed a pattern in the function $f(10^n)$.

$$f(10^1) = 2 = 1 + 1$$
$$f(10^2) = 21 = 20 + 1$$
$$f(10^3) = 301 = 300 + 1$$
$$f(10^4) = 4001 = 4000 + 1$$
$$f(10^5) = 50001 = 50000 + 1$$

The pattern is $f(10^n) = n10^{n-1} + 1$.

Therefore, $f(10^{100}) = 100(10^{99}) + 1 = 10^{101} + 1$.

Comment from the solutions editor: It would have been more elegant for Mr. Walton to calculate $f(10^{100})$ directly with his combinatorial method, as

$$f(10^{100}) = 1\binom{100}{1}9^{99} + 2\binom{100}{2}9^{98} + 3\binom{100}{3}9^{97} + \ldots + 99\binom{100}{99}9^{1} + 100\binom{100}{100}9^{0} + 1,$$

but since that would require calculating and summing 100 numbers, most of them very large, we understand that he had to resort to a shortcut.

We can derive Mr. Walton's pattern from his combinatorial method by using more advanced combinatorics. Mr. Walton's method gives that

$$f(10^n) = 1 + \sum_{i=1}^{n} i\binom{n}{i}9^{n-i}.$$

Consider the product $i\binom{n}{i}$.

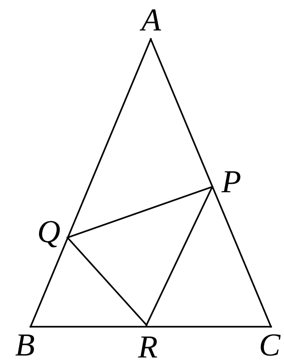
$$i\binom{n}{i} = i \times \frac{n!}{i!(n-i)!} = \frac{n!}{(i-1)!(n-i)!} = n \times \frac{(n-1)!}{(i-1)!(n-i)!} = n\binom{n-1}{i-1}$$

So

$$f(10^n) = 1 + \sum_{i=1}^{n} n\binom{n-1}{i-1}9^{n-i} = 1 + n\sum_{j=0}^{n-1}\binom{n-1}{j}(1)^j(9)^{(n-1)-j} = 1 + n(1+9)^{n-1}$$

where $j = i - 1$ and the last step is by the binomial theorem. Thus, $f(10^n) = 1 + n10^{n-1}$.

5/2/14. For an isosceles triangle ABC where $AB = AC$, it is possible to construct, using only compass and straightedge, an isosceles triangle PQR where $PQ = PR$ such that triangle PQR is similar to triangle ABC, point P is in the interior of line segment $\overline{AC}$, point Q is in the interior of line segment $\overline{AB}$, and point R is in the interior of line segment $\overline{BC}$. Describe one method of performing such a construction. Your method should work on every isosceles triangle ABC, except that you may choose an upper limit or lower limit on the size of angle BAC.



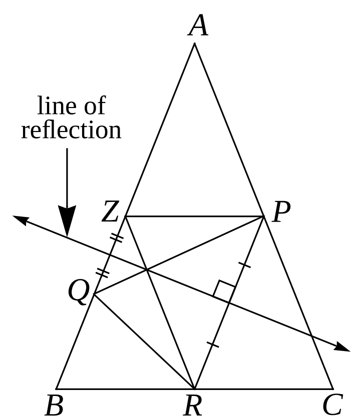
Comments: This problem was devised by George Berzsenyi, the founder and writer of the USAMTS.

The graders of this problem noticed that few of the submitted constructions bothered being precise about the upper limit on the size of $\angle BAC$ (no construction needed a lower limit). Rather than calculating the largest angle for which the construction would work, most participants merely declared that the angle was acute, which did fit the letter of the problem. Thus, the graders decided that the precise upper limit was worth zero points.

Solution 1 to 5/2/14 by Eline Boghaert (11/NY):

1. In order for this method of construction to work, the size of $\angle BAC$ should be less than 90˚.

2. Find the midpoints of $\overline{AC}$, $\overline{BC}$, and $\overline{AB}$, and label them P, R, and Z, respectively. (To draw the midpoint of $\overline{AC}$, set your compass so that it is wider than half of $\overline{AC}$'s length, such as setting it to $\overline{AC}$'s length. Draw a circle centered at point A. Using the same compass radius, draw a circle centered at point C. These circles intersect at two points: connect these points of intersection with a line segment. This line segment is the perpendicular bisector of $\overline{AC}$; the midpoint of $\overline{AC}$ is where the perpendicular bisector intersects $\overline{AC}$. Construct midpoints for $\overline{BC}$ and $\overline{AB}$ by the same method.) Connect the midpoints P, R, and Z with line segments. A triangle RPZ similar to the larger triangle ABC is created, with $\angle PRZ$ congruent to $\angle BAC$.



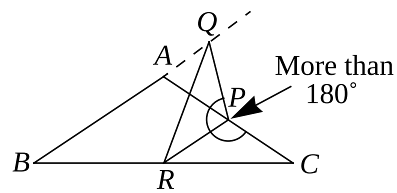
3. Draw the perpendicular bisector to $\overline{PR}$ (same process as before). Since the line segment connecting any two midpoints of sides of a triangle is parallel to the side it does not intersect, $\overline{PR}$ is parallel to $\overline{AB}$. Thus, the perpendicular bisector of $\overline{PR}$ is also perpendicular to $\overline{AB}$.

4. Reflect ΔRPZ through the perpendicular bisector of $\overline{PR}$. $\overline{PR}$ reflects onto itself. To reflect point Z, copy the distance between Z and line of reflection to the other side of the line of reflection. Since $\overline{AB}$ is perpendicular to the line of reflection, point Z's reflection, call it Q,

will also be on $\overline{AB}$. Draw the line segments $\overline{PQ}$ and $\overline{RQ}$ to complete the triangle PQR. Since ΔPQR is a reflection of ΔRPZ, it is similar to ΔABC, with $\angle QPR$ congruent to $\angle BAC$.

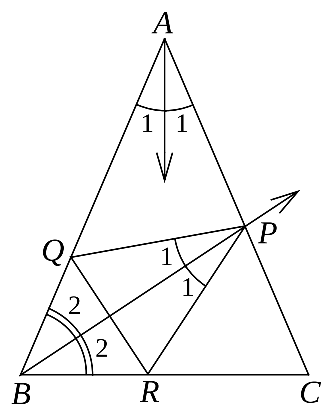
Comment from the solutions editor: Given points R and Z in Ms. Boghaert's construction, there are many ways to find point Q. For example, $RQ = RB$, so we could draw a circle of radius RB centered at R. Many participants found different ways to construct this triangle.

If the size of $\angle BAC$ were more than 90˚, then $\angle QPR$ and $\angle RPC$ together would measure more than 180˚, so point Q, though still on line $\overleftrightarrow{AB}$, would be on the wrong side of A.



Solution 2 for 5/2/14 by Anton Kriksunov (11/NY):

To construct the triangle, I first made a bisection of angle ABC and another bisection of angle BAC. Point P is the intersection of the bisector of angle ABC and the side $\overline{AC}$. Next at point P I copied half of angle BAC onto both sides of $\overline{BP}$, creating angles QPB and RPB, where Q is where the upper ray intersects $\overline{AB}$ and R is where the lower ray intersects $\overline{BC}$. Then I drew line segment $\overline{QR}$ to finish triangle PQR. It is similar to triangle ABC because both are isosceles and have congruent vertex angles, which means all their angles are congruent.

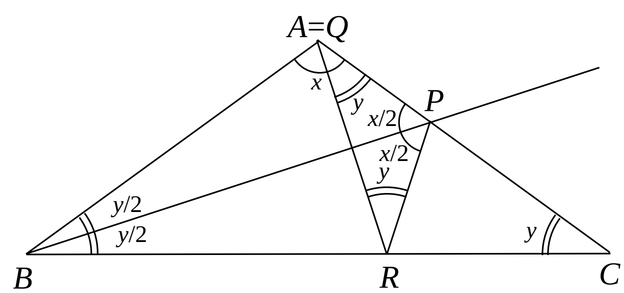


This method works well for all acute and right isosceles triangles, but there is a problem with obtuse ones. After their vertex angles reach a certain amount, the constructed triangle is no longer contained within the original triangle, so the construction does not fulfill the conditions of the problem.

This similar triangle is just on the border of the original triangle: if its vertex angle increases any more, point Q will move outside the original triangle. This then is the limit for the size of angle BAC. To find the angle, I set up two equations:

$x + y + y = 180°$ from triangle ABC, and

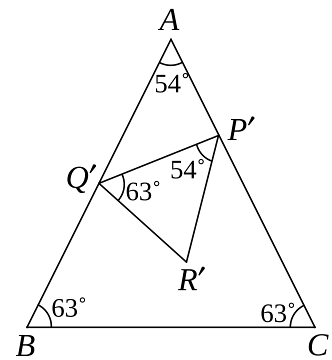
$x + y/2 + x/2 = 180°$ from triangle ABP.



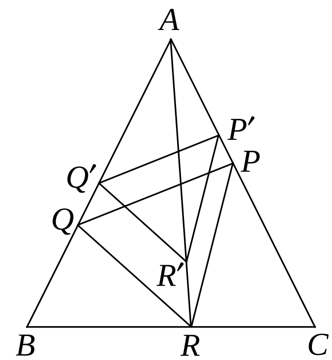
Solving gives $x = 108°$ and $y = 36°$. So the restrictions on angle BAC are that $0° < m\angle BAC < 108°$.

Solution 3 for 5/2/14:

From the solutions editor: Another method for constructing triangle PQR is by constructing a triangle that is the right shape and right orientation and rescaling it to the proper size by projection. Alas, the example of this method selected by the graders did not have the solver's signature on the cover sheet giving permission to use his name. It would be improper to use his solution without giving credit. So I will write my own version of a projection construction.



Draw a point Q' anywhere on the interior of side $\overline{AB}$. Draw a point P' anywhere on the interior of side $\overline{AC}$ except that $\angle BQ'P'$ must be larger than $\angle ABC$. Locate point R' in the interior of $\angle BQ'P'$ as follows. We draw a ray from point Q' so that the angle between the ray and $\overline{Q'P'}$ copies $\angle ABC$ and draw another ray from point P' so that the angle between that ray and $\overline{Q'P'}$ copies $\angle BAC$. Point R' is the intersection of those two rays.



Because $\angle BQ'P'$ is larger than $\angle R'Q'P'$ (which is congruent to $\angle ABC$), ray $\overrightarrow{Q'R'}$ is in the interior of $\angle BQ'P'$. We can also show that $\angle CP'R'$ is congruent to angle $\angle AQ'P'$, so ray $\overrightarrow{P'R'}$ is in the interior of $\angle CP'Q'$. Therefore, point R' is in the interior of $\angle BAC$.

If point R' happens to be on side $\overline{BC}$, we are done, but more likely it is above or below that side. In that case, draw ray $\overrightarrow{AR'}$. Since the ray is in the interior of $\angle BAC$, it intersects side $\overline{BC}$, in the interior of that side. Call the point of intersection R. Draw a line through point R that is parallel to $\overline{Q'R'}$. Let Q be the point where that line and line $\overleftrightarrow{AB}$ intersect. Draw a line through point R that is parallel to $P'R'$. Let P be the point where that line and line $\overleftrightarrow{AC}$ intersect. Draw line segment $\overline{QP}$.

Triangle PQR is similar to triangle $P'Q'R'$ because it is a projection about point A of triangle $P'Q'R'$. Thus, it is similar to triangle ABC. For those unfamilar with the properties of projection, note that triangle AQR is similar to triangle $AQ'R'$ and triangle APR is similar to triangle $AP'R'$, both in the same ratio of dimensions. Since $\angle QRP \cong \angle Q'R'P'$ and $PR/QR = P'R'/Q'R'$, the triangles PQR and $P'Q'R'$ are similar.

Because $\angle AQR$ is larger than $\angle ABC$, point R is lower than point Q when viewing side $\overline{BC}$ as the base of triangle ABC. So Q is above side $\overline{BC}$ and below point A, so it is in the interior of side $\overline{AB}$. Tracing angles around the triangle gives that $\angle APR$ is congruent to $\angle BQP$, which is larger than $\angle ACB$, so P is likewise in the interior of side $\overline{AC}$.

This construction works for all triangles, regardless of the size of $\angle BAC$.