USA Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 4 - Year 12 - Academic Year 2000-2001

Gene A. Berg, Editor

1/4/12. Determine all positive integers with the property that they are one more than the sum of the squares of their digits in base 10.

Solution 1 by Steve Byrnes (10/MA): Let the number be, in base ten, $...a_5a_4a_3a_2a_1a_0$, where any of the a's may be zero. Then the condition is equivalent to

$$1 + a_0^2 + a_1^2 + a_2^2 + a_3^2 + a_4^2 + \ldots = a_0 + a_1 10^1 + a_2 10^2 + a_3 10^3 + a_4 10^4 + \ldots,$$

or

$$0 = 1 + a_0(a_0 - 1) + a_1(a_1 - 10) + a_2(a_2 - 100) + a_3(a_3 - 1000) + a_4(a_4 - 10000) + \ldots$$

$$a_1(a_1 - 10) + a_2(a_2 - 100) + a_3(a_3 - 1000) + \ldots = -1 - a_0(a_0 - 1) \geq -1 - 9 \times 8 = -73.$$

All terms on the left side are negative, since the a_i are integers 0 to 9. Also, after the leftmost term, the terms are all zero or smaller that -73. Hence, we must conclude that

$$0 = a_2 = a_3 = a_4 = a_5 = \ldots.$$

Therefore $$a_0(a_0 - 1) + 1 = a_1(10 - a_1).$$

As a_0 goes from 0 to 9, the left side can be 1, 3, 7, 13, 21, 31, 43, 57, or 73. As a_1 goes from 0 to 9, the right side can be 0, 9, 16, 21, 24, 25, 21, 16, 9, or 0. The only number these lists have in common is 21, so $a_0 = 5$ and $a_1 = 3$ or 7. Hence, the two possible numbers are **35 and 75**.

Solution 2 by Yuen-Joyce Liu (9/MA): **35** and **75** are the only two positive integers with the property that they are one more than the sum of the squares of their digits.

Table 1:

Number of digits in integer (n)	Maximal integer represented by k digits (10^{n-1})	maximal sum of the squares of k digits plus 1 (9^2n+1)
1	1	82
2	10	163
3	100	244
4	1000	325
5	10000	406

Table 1 suggests that the largest possible number of digits in an integer which satisfies the problem requirement is 3. We use mathematical induction to prove

Claim: $10^{n-1} > 9^2 n + 1 = 81n + 1$ for integer $n > 3$.

Proof of Claim: For $n = 4$, $10^{4-1} = 1000 > 81 \times 4 + 1 = 325$.

Assume for $n = k > 3$, that $10^{k-1} > 81k + 1$. Then for $n = k + 1$ we have

$$\begin{aligned} 10^{(k+1)-1} &= 10 \times 10^{k-1} \\ &> 10(81k + 1) \\ &= 810k + 10 \\ &= 81(k + 1) + (729k - 71) \\ &> 81(k + 1) + 1 \end{aligned}$$

Therefore the Claim is proved, and an integer which satisfies the problem requirement can have at most 3 digits.

The maximal sum of squares of three digits plus one is 244. So there can be no integer n meeting the requirements with $n > 244$. The largest possible integer of three digits with 1 or 2 in the hundreds digit is 299, whose sum of the square of its digits plus one is 166, so there can be no integer n meeting the requirements with $n > 166$. For $100 \le n \le 166$, the value of n which has largest sum of its squares plus one is n = 159, whose sum of squares plus one equals 107. So there can be no integer n meeting the requirements with $n > 107$. For $100 \le n \le 107$, the value of n which has largest sum of its squares plus one is n = 107, whose sum of squares plus one equals 50. So there can be no integer n meeting the requirements with $n > 99$. So the largest possible number of digits in an integer which satisfies the requirement is reduced to 2.

In order for a positive integer with the tens digit a and the units digit b to satisfy the problem requirement, we have

$$10a + b = 1 + a^2 + b^2$$

$$a^2 - 10a + (b^2 - b + 1) = 0$$

$$a = \frac{10 \pm \sqrt{100 - 4(b^2 - b + 1)}}{2}$$

Table 2:

b	0	1	2	3	4	5	6	7	8	9
$\sqrt{100 - 4(b^2 - b + 1)} \approx$	9.80	9.80	9.38	8.49	6.93	4	Unreal	Unreal	Unreal	Unreal
a	...	...	...	...	...	7 or 3	...	...	...	...

As calculated and shown in Table 2, we find that **35** and **75** are the only two positive integers with the property that they are one more than the sum of the squares of their digits in base 10.

1/4/12. Editor's Comment: We are grateful to Professor Bruce Reznick of the University of Illinois for communicating this problem to us. He created this problem for the 1990 Friendly Competition (The Indiana College Mathematics Competition).

2/4/12. Prove that if n is an odd positive integer, then $N = 2269^n + 1779^n + 1730^n - 1776^n$ is an integer multiple of 2001.

Solution 1 by Anatoly Preygel (10/MD): Note that 2001 factors into primes as $3 \cdot 23 \cdot 29$. Thus it is sufficient that $3|N$, $23|N$, and $29|N$.

Note that for all integers n, $a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + \ldots + a \cdot b^{n-2} + b^{n-1})$ and for odd integer n, $a^n + b^n = (a+b)(a^{n-1} - a^{n-2}b + \ldots - a \cdot b^{n-2} + b^{n-1})$. Thus, for odd n, $(a+b)|(a^n + b^n)$ and $(a-b)|(a^n - b^n)$.

In this case, note that $N = 2269^n + 1779^n + 1730^n - 1776^n$ and that

$$(2269 + 1779)|(2269^n + 1779^n)$$
$$2269 + 1779 = 4048 = 23 \cdot 176$$
$$(1730 - 1776)|(1730^n + 1776^n)$$
$$1730 - 1776 = -46 = 23 \cdot (-2)$$

Thus, $23|N$.

Now, note that $N = 2269^n - 1776^n + 1779^n + 1730^n$ (just reordering terms), and that

$$(2269 - 1776)|(2269^n + 1776^n)$$
$$2269 - 1776 = 493 = 29 \cdot 17$$
$$(1779 + 1730)|(1779^n + 1730^n)$$
$$1779 + 1730 = 3509 = 29 \cdot 121$$

Thus, $29|N$.

Last, note that $1779 = 3 \cdot 593$, $1776 = 3 \cdot 592$, and that

$$(2269 + 1730)|(2269^n + 1730^n)$$
$$2269 + 1730 = 3999 = 3 \cdot 1333$$

Thus $3|N$.

Thus we have $2001|N = 2269^n + 1779^n + 1730^n - 1776^n$ for odd integer n.

Solution 2 by Valerie Lee (10/NY):

$$N = 2269^n + 1779^n + 1730^n - 1776^n$$

N is multiple of 2001?: $\quad 2001x = 2269^n + 1779^n + 1730^n - 1776^n$.

Factor 2001: $\quad 3 \cdot 23 \cdot 29 \cdot x = 2269^n + 1779^n + 1730^n - 1776^n$.

If N is divisible by the factors of 2001, then it has to be divisible by 2001. Writing N modulo these factors, the expressions should be equal to zero. If this is true, then N is a multiple of 2001.

Consider mod 29:	$2269^n + 1779^n + 1730^n - 1776^n \pmod{29}$
Replace terms by remainders mod 29.	$\equiv 7^n + 10^n + 19^n - 7^n \pmod{29}$
$19^n \equiv (-10)^n \pmod{29}$	$\equiv 7^n + 10^n + (-10)^n - 7^n \pmod{29}$
Everything cancels if n is odd	$\equiv 0$

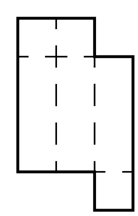
Continue mod 23:	$2269^n + 1779^n + 1730^n - 1776^n \pmod{23}$
Replace terms by remainders mod 23.	$\equiv 15^n + 8^n + 5^n - 5^n \pmod{23}$
$15^n \equiv (-8)^n \pmod{23}$	$\equiv (-8)^n + 8^n + 5^n - 5^n \pmod{23}$
Everything cancels if n is odd	$\equiv 0$

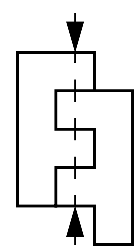
Last mod 3:	$2269^n + 1779^n + 1730^n - 1776^n \pmod{3}$
Replace terms by remainders mod 3.	$\equiv 1^n + 0^n + 2^n - 0^n \pmod{3}$
$2^n \equiv (-1)^n \pmod{3}$	$\equiv 1^n + 0^n + (-1)^n - 0^n \pmod{3}$
Everything cancels if n is odd	$\equiv 0$

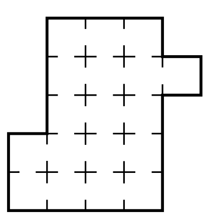
Thus, N is divisible by 2001 when n is odd.

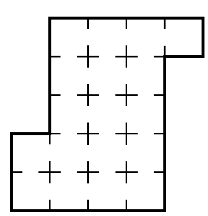
Editor's Comment: We thank our problem editor, Dr. George Berzsenyi, for this timely problem.

3/4/12. The figure on the right can be divided into two congruent halves that are related to each other by a glide reflection, as shown below it. A glide reflection reflects a figure about a line, but also moves the reflected figure in a direction parallel to that line. For a square-grid figure, the only lines of reflection that keep its reflection on the grid are horizontal, vertical, 45° diagonal, and 135° diagonal. Of the two figures below, divide one figure into two congruent halves related by a glide reflection, and tell why the other figure cannot be divided like that.

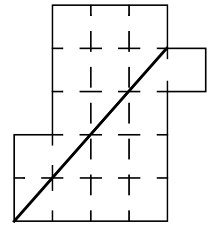




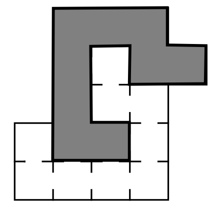




Solution 1 by Katherine Herbig (11/WA): The figure must slide along a line parallel to the line it is reflected across. Therefore, the figure and its reflection must have the same number of squares that either cut the line or share an edge with it. Therefore, the line of reflection must cut through an even number of squares, or touch the same number of edges on each side of the line.



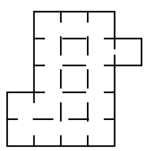
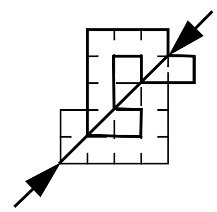
In addition, there must be the same number of squares on each side of the line because all squares on one side of the line at the beginning of a glide reflection will be on the other side after the glide reflection.



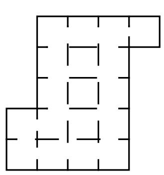
There is only one line in either of the squares that fits both requirements. It is shown in the top diagram on the right. The division is shown in the lower diagram.

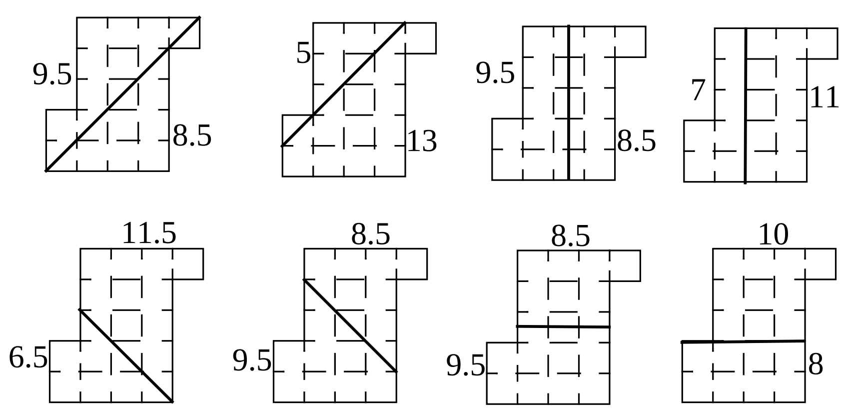
The other figure cannot be divided by a glide reflection because it has only one line that is even near the middle of the figure. This line crosses through an even number of squares, but does not divide the figure evenly.

Solution 2 by Katheryn Green (11/WV):

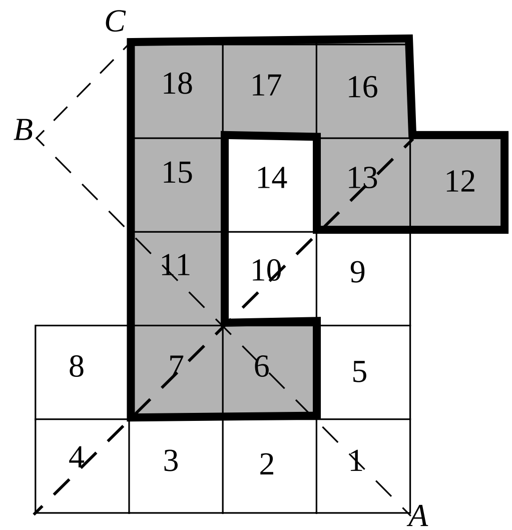
The figure  can be divided using the $45°$ glide reflection: 

For a glide reflection to be possible, the reflecting line must divide the figure so that there is an equal number of squares on each side. Notice that there is a total of six squares on each side of the line in the example, and a total of nine on each side in the solution above. The figure at left below cannot be divided into two congruent halves related by a glide reflection because, as shown, it is impossible to divide the figure in half with a horizontal, vertical, $45°$, or $135°$ line that lies along the grid lines.





Solution 3 by Igor Zhitnitskiy (10/NY): It is given that the line of reflection can only be horizontal, vertical, or diagonal at 45 degrees in either direction. Thus for a given figure, there is a maximum of four lines of reflection. However, a line of reflection must also have an equal area on either side (since a glide reflection only moves the reflected portion parallel to the line and does not change its area). It must also go through corners or midpoints of lines on the square grid; it cannot go through any other point because a reflection would not be possible. Keeping this in mind, it is simple to verify, by trial and error, that only one line of reflection exists for the figure at right (expressed as a broken line from the corner of block 4 to the corner of block 13). The other figure has no such lines of reflection that maintain equal areas while intersecting only corners and midpoints; thus any kind of glide reflection is impossible.

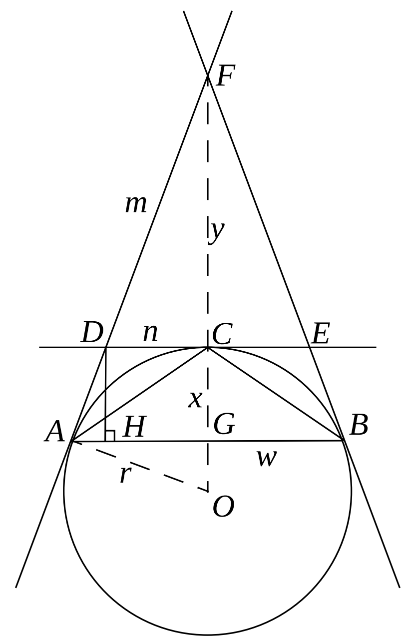


Once the line of reflection is identified the procedure for determining the congruent halves is as follows. Since blocks 1 and 18 are the only two blocks with a distance of 2 diagonals from their distant corners to the line of reflection, they must be corresponding parts of the two congruent halves. Let us denote blocks associated with **block 1 as "white"** and the blocks associated with **block 18 as "gray"**, and the side of reflection on which block 18 is located as the "gray" side, with the other side of the reflection line as the white side. From this relationship it is clear that blocks on the white side are related to their corresponding blocks on the gray side by a reflection and 1 block movement "up" along the line of reflection (as in the diagram $A \rightarrow B \rightarrow C$). If block 12 were white, then its glide reflection would be 1 block along the axis away from block 16. However there is no block there. Thus **block 12 must be gray**. There must then be a white block reflected across and 1 block "down" from 12. This is block 14; thus **block 14 is white**. These two objects must have their blocks connected to form one figure, so blocks 18 and 12 must be "bridged" by either blocks 17, 16, and 13, or blocks 15, 11, 10, 9,and 13. However if the later is used, block 14 is isolated from block 1. Thus **blocks 17, 16,and 13 are gray**. The corresponding blocks for these are white, so **blocks 5, 9, and 10 are white**. If block 8 is gray, then there must be a white block one block "down" from block 3 along the axis. There is no block there, so block **8 must be white**, and its counterpart, **block 6, is gray**. 6 must not be isolated from the gray figure; thus **15, 11, and 7 must be gray**. Their respective counterparts **2, 3, and 4 must be white**. As stated earlier the other given figure has no applicable lines of reflection, so this is the one and only glide reflection possible for the two figures.

Editor's Comment: We are indebted to Dr. Erin Schram of the National Security Agency for formulating this problem, and to Professor Kimmo Eriksson of Sweden, whose article "Splitting a Polygon into Two Congruent Pieces" in the May 1996 issue of The American Mathematical Monthly served as an inspiration for this problem.

4/4/12. Let A and B be points on a circle which are not diametrically opposite, and let C be the midpoint of the smaller arc between A and B. Let D, E, and F be the points determined by the intersections of the tangent lines to the circle at A, B, and C. Prove that the area of ΔDEF is greater than half of the area of ΔABC.

Solution 1 by Rishi Gupta (8/CA): Let O be the center of the circle, with r as the radius. FO splits the diagram in half, so one side is symmetric to the other.



First, I saw that $AF > w$ because of right ΔGAF. $AF = m + n$, as shown in the diagram, so $m + n > w$. Now $DA \cong DC$ (because both are tangents), so we can replace our equation with $m + DC > w$. Because of right ΔCDF, $m > DC$. Therefore, $2m > w$.

Then, I wanted to prove that $\frac{x}{n} = \frac{y}{m}$. I drew a perpendicular line segment down from D to point H on AB, so that $DH = x$. Now $\Delta HAD \sim \Delta CDF$ because both are right triangles and $\angle DAG \cong \angle FDC$ $(DE || AB)$, Therefore, $\frac{HD}{DA} = \frac{CF}{FD}$, or $\frac{x}{n} = \frac{y}{m}$.

Going back to my previous inequality, $2m > w$, we have:

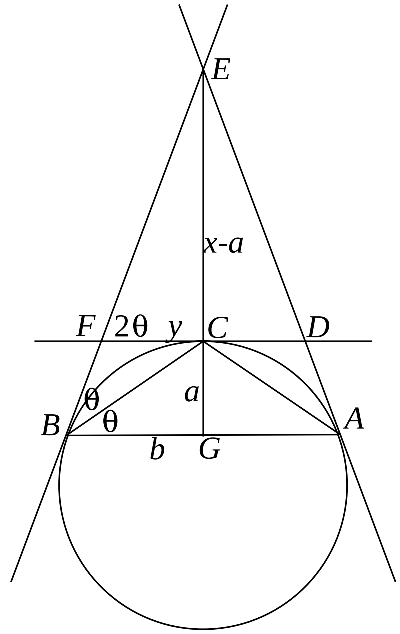
$$2m > w$$

$$2m \cdot \left(\frac{y}{m}\right) > w \cdot \left(\frac{x}{n}\right)$$

$$ny > \frac{wx}{2}.$$

The area of $\Delta DEF = ny$ and the area of $\Delta ABC = wx$. Substituting, we have the area of ΔDEF is greater than half the area of ΔABC.

Solution 2 by Nathaniel Jones (12/IA): In the figure at the right, let $CG = a$, $BG = b$, $EG = x$, and $CF = y$. Since $m\overset{\frown}{AC} = m\overset{\frown}{BC}$, $m\angle ABF = 2m\angle CBF$. Let $m\angle ABC = m\angle CBF = \theta$. Using trigonometry, $\tan\theta = \frac{a}{b}$, and



$$\tan 2\theta = \frac{\tan\theta}{1-\tan^2\theta} = \frac{2\left(\frac{a}{b}\right)}{1-\left(\frac{a^2}{b^2}\right)} = \frac{2ab}{b^2-a^2} = \frac{x}{b}.$$

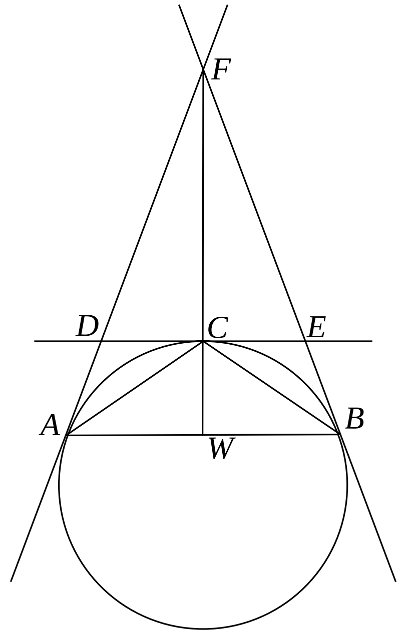
Thus, $x = \frac{2ab^2}{b^2-a^2}$. Since both *AB* and *DF* are perpendicular to *EG*, $m\angle ABE = m\angle DFE$. This means that $\tan 2\theta = \frac{2ab}{b^2-a^2} = \frac{x-a}{y} = \frac{\left(\frac{2ab^2}{b^2-a^2}\right)-a}{y}$.

Combining this with the expression above for $\tan 2\theta$ gives $y = \frac{a^2+b^2}{2b}$.

The area of ΔABC is $\frac{2ab}{2} = ab$, and

the area of ΔDEF is $2\times\frac{1}{2}\left(\frac{a^2+b^2}{2b}\right)\left(\frac{2ab^2}{b^2-a^2}-a\right) = \frac{a^5+2a^3b^2+ab^4}{2(b^3-a^2b)}$. To show that the area of ΔDEF is greater than half the area of ΔABC, it must be proven that $\frac{1}{2}ab < \frac{a^5+2a^3b^2+ab^4}{2(b^3-a^2b)}$, or that $0 < \frac{a^5+2a^3b^2+ab^4}{2(b^3-a^2b)} - \frac{1}{2}ab$. This statement simplifies to $0 < a^4+3a^2b^2$, and since both *a* and *b* are always positive, the statement is always true. So the area of ΔDEF is greater than half the area of ΔABC.

Solution 3 by Kevin Yang (11/IL): Let *W* be the midpoint of line segment *AB*. Line segment *AD* is congruent to *DC* because of the two-tangent rule. Now, we know that *AC* is longer than AW, because the hypotenuse of any right triangle is longer than either leg. Also, the lengths of *AD* plus that of *DC* will be greater than the length of *AC* because of the triangle inequality. So, the length of *AD* plus that of *DC* will be greater than the length of *AW*. And thus, *DC* will be longer than half the length of *AW*.

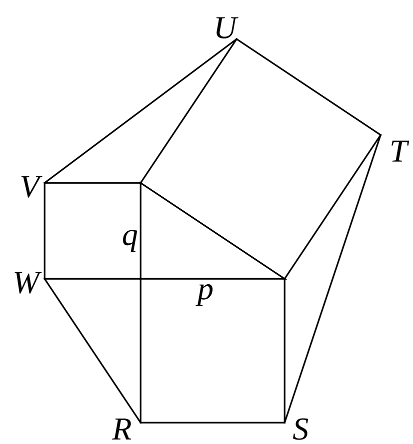


From this we know that *DE* is more than half the length of *AB*. Since the triangles *DFE* and *ABF* are similar (because *AB* is parallel to *DE*), we also know that *FC* is more than half the length of *FW*. In other words, *FC* is longer than *CW*. **The area of a triangle is (1/2)(base)(height), so since the base of triangle *DEF* is more than half that of triangle *ABC*, and its height greater than that of triangle ABC, then the area of triangle DEF must be more than half the area of triangle ABC.**

Editor's comment: This historically interesting problem was proposed by Dr. Peter Anspach of NSA. He found it in a paper by mathematician/astronomer Christiaan Huyghens, who used it as a stepping stone for calculating the digits of pi.

5/4/12. Hexagon *RSTUVW* is constructed by starting with a right triangle of legs measuring *p* and *q*, constructing squares outwardly on the sides of this triangle, and then connecting the outer vertices of the squares, as shown in the figure on the right.

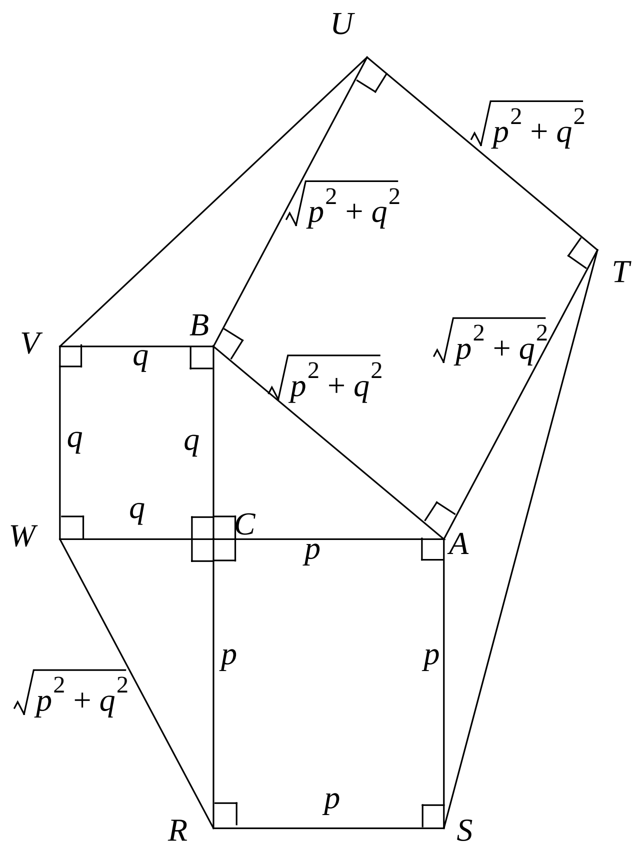
Given that *p* and *q* are integers with $p > q$, and that the area of *RSTUVW* is 1922, determine *p* and *q*.



Solution 1 by Laura Pruitt (11/MA):

From the given information we can determine the following segment lengths, areas, and angles:

$$\alpha(ABUT) = \sqrt{p^2+q^2}$$
$$\alpha(ACRS) = p^2$$
$$\alpha(BCWV) = q^2$$
$$\alpha(\Delta ABC) = \frac{pq}{2}$$
$$\alpha(\Delta CRW) = \frac{pq}{2}$$



We now need only the areas of ΔBUV and ΔAST.

$m(\angle VBU) + m(\angle CBA) = 180°$ so

$$\sin(\angle VBU) = \sin(\angle CBA) = \frac{p}{\sqrt{p^2+q^2}}$$

Similarly

$$\sin(\angle TAS) = \sin(\angle BAC) = \frac{q}{\sqrt{p^2+q^2}}$$

So

$$\alpha(\Delta BUV) = \left(\frac{1}{2}\right)q\sqrt{p^2+q^2}\sin(\angle VBU) = \left(\frac{1}{2}\right)q\sqrt{p^2+q^2}\left(\frac{p}{\sqrt{p^2+q^2}}\right) = \frac{pq}{2}$$

$$\alpha(\Delta AST) = \left(\frac{1}{2}\right)p\sqrt{p^2+q^2}\sin(\angle TAS) = \left(\frac{1}{2}\right)q\sqrt{p^2+q^2}\left(\frac{q}{\sqrt{p^2+q^2}}\right) = \frac{pq}{2}$$

We now know the areas of all the pieces of the hexagon in terms of *p* and *q*. So

$$1922 = \alpha(ABUT) + \alpha(ACRS) + \alpha(BCWV) + \alpha(\Delta ABC) + \alpha(\Delta CRW) + \alpha(\Delta BUV) + \alpha(\Delta AST)$$
$$= p^2+q^2+p^2+q^2+\frac{pq}{2}+\frac{pq}{2}+\frac{pq}{2}+\frac{pq}{2} = 2p^2+2q^2+2pq$$

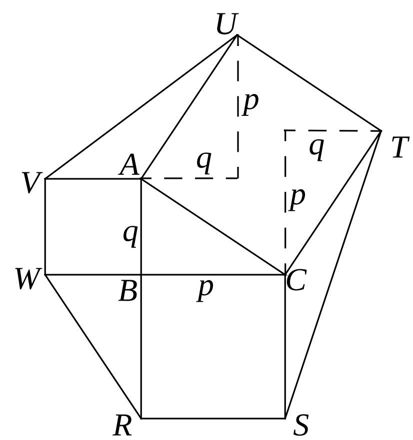
So $961 = p^2+pq+q^2$. The quadratic equation gives $p = \frac{-q \pm \sqrt{q^2-4(q^2-961)}}{2}$. Looking at the determinant, $q^2-4(q^2-961) = 3844-3q^2$, we see that, since the discriminant must be greater than or equal to zero, since $q < p$, and since q is an integer, it follows that $1 \le q \le 17$. Furthermore, from this range we can see that the square root of the discriminant must be between 55 and 61 inclusive. Call this value *s*.

$\frac{3844 - s^2}{3} = q^2$, which must be a perfect square. Of the range 55 to 61, only one value of s works: s = 59. $\frac{3844 - 3481^2}{3} = 121$ giving $q = 11$.

This gives $p = \frac{-11 \pm \sqrt{11^2 - 4(11^2 - 961)}}{2} = 24$.

So p = 24, q = 11.

Solution 2 by Ho Seung (Paul) Ryu (09/KS): To make referring to the regions easier, label the vertices of the triangle A, B, and C. [ABC], the notation meaning the area of ABC, is obviously $\frac{pq}{2}$. We also have



[$ABWV$] = q^2,
[$BCSR$] = p^2
[$ACTU$] = $p^2 + q^2$ from the Pythagorean Theorem.

Additionally, by looking at the diagram, we also see that [VAU] = [CST] = $\frac{pq}{2}$, so [$RSTUVW$] is the sum of all the individual areas, which sum to $2p^2 + 2pq + 2q^2$. So we now have that $1922 = 2p^2 + 2pq + 2q^2$, or simply $961 = p^2 + pq + q^2 = (p+q)^2 - pq$, or $961 + pq = (p+q)^2$. Since $961 = 31^2$, $p + q$ is greater than 31.

If $p + q$ is 32, pq must equal 63. No such combination exists. The same occurs for $p + q$ equal to 33 and 34. But if $p + q = 35$, then $(p, q) = (24, 11)$ satisfies the inequality.

So if **(p, q) = (24, 11)**, then the area of $RSTUVW$ is 1922.

Editor's comments: This problem was inspired by Problem 54 in *Quantum Quandaries*, published in 1996 by the National Science Teachers Association. We thank Dr. Berzsenyi for posing this problem.