U S A Mathematical Talent Search

PROBLEMS / SOLUTIONS / COMMENTS
Round 4 - Year 10 - Academic Year 1998-99

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1/4/10. Exhibit a 13-digit integer *N* that is an integer multiple of 2^{13} and whose digits consist of only 8s and 9s.

Solution 1 by Ann Marie Cody (12/MA): Let N = ABCDEFGHIJKLM, where A, B, ..., M are digits.

N is divisible by 2, so M must be even. Therefore M = 8.

N = ABCDEFGHIJKx100 + LM and is divisible by 2^{13}. N and ABCDEFGHIJKx100 are each divisible by 4, so LM must be divisible by 4. LM = L8, 88 is divisible by 4, and 98 is not, so LM = 88.

N = ABCDEFGHIJx1000 + KLM is divisible by 8 and 1000 is divisible by 8, so KLM must be divisible by 8. 8 does not divide 988, so KLM = 888.

N = ABCDEFGHIx10,000 + JKLM. N and 10,000 are divisible by 2^4, so JKLM is also. JKLM = J888. 8888 is not divisible by 16 but 9888 is, so JKLM = 9888.

N = ABCDEFGHx100,000 + IJKLM. N and 100,000 are divisible by 2^5, so IJKLM is also. IJKLM = I9888. 99888 is not divisible by 32 but 89888 is, so IJKLM = 89888.

N = ABCDEFGx1,000,000 + HIJKLM. N and 1,000,000 are divisible by 2^6, so HIJKLM is also. HIJKLM = H89888. 889888 is not divisible by 64 but 989888 is, so HIJKLM = 989888.

N = ABCDEFx10,000,000 + GHIJKLM. N and 10,000,000 are divisible by 2^7, so GHIJKLM is also. GHIJKLM = G989888. 8989888 is not divisible by 128 but 9989888 is, so GHIJKLM = 9989888.

N = ABCDEx100,000,000 + FGHIJKLM. N and 100,000,000 are divisible by 2^8, so FGHIJKLM is also. FGHIJKLM = F9989888. 99989888 is not divisible by 256 but 89989888 is, so FGHIJKLM = 89989888.

N = ABCDx10^9 + EFGHIJKLM. N and 10^9 are divisible by 2^9, so EFGHIJKLM is also. EFGHIJKLM = E89989888. 889989888 is not divisible by 512 but 989989888 is, so EFGHIJKLM = 989989888.

N = ABCx10^{10} + DEFGHIJKLM. N and 10^{10} are divisible by 2^{10}, so DEFGHIJKLM is also. DEFGHIJKLM = D989989888. 9989989888 is not divisible by 2^{10} but 8989989888 is, so DEFGHIJKLM = 8989989888.

N = ABx10^{11} + CDEFGHIJKLM. N and 10^{11} are divisible by 2^{11}, so CDEFGHIJKLM is also. CDEFGHIJKLM = C8989989888. 88989989888 is not divisible by 2^{11} but 98989989888 is, so CDEFGHIJKLM = 98989989888.

N = Ax10^{12} + BCDEFGHIJKLM. N and 10^{12} are divisible by 2^{12}, so BCDEFGHIJKLM is also. BCDEFGHIJKLM = B98989989888. 998989989888 is not divisible by 2^{12} but 898989989888 is, so BCDEFGHIJKLM = 898989989888.

N and 10^{13} are divisible by 2^{13}. N=ABCDEFGHIJKLM = A98989989888. 9898989989888 is not divisible by 2^{13} but 8898989989888 is, so

N = 8898989989888

Solution 2 by Oaz Nir (10/CA): Answer: 8898989989888

Proof: More generally, we prove that for all positive integers n and non-zero digits a and b, with a even and b odd, there exists an integer x_n with exactly n digits that is an integer multiple of 2^n, whose digits consist of only a's and b's. The proof is by mathematical induction. For the base case we take $x_1 = a$. Clearly this number is divisible by $2^1 = 2$, as required. For the induction step, we assume that an integer x_n exists with the desired properties, and we show how to construct x_{n+1}. We need two cases:

Case 1: x_n is divisible by 2^{n+1}.

Take x_{n+1} equal to x_n with the digit a appended to the left. That is, $x_{n+1} = a \cdot 10^n + x_n$.

It is easy to see that x_{n+1} is divisible by 2^{n+1} (because both terms $a \cdot 10^n$ and x_n are).

Case 2: x_n is not divisible by 2^{n+1}.

Take x_{n+1} equal to x_n with the digit b appended to the left. That is, $x_{n+1} = b \cdot 10^n + x_n$.

It is easy to see that x_{n+1} is divisible by 2^{n+1} (because both terms $b \cdot 10^n$ and x_n are odd multiples of 2^n, so their sum is an even multiple of 2^n, and is therefore a multiple of 2^{n+1} as required).

This completes the proof of the general result. We can use our method of proof to actually calculate the numbers x_n. In particular, when $a = 8, b = 9,$ and $n = 13,$ we proceed as follows: $x_1 = 8,$ $x_2 = 88,$ $x_3 = 888,$ $x_4 = 9888,$ $x_5 = 89888,$ $x_6 = 989888,$ $x_7 = 9989888,$ $x_8 = 89989888,$ $x_9 = 989989888,$ $x_{10} = 8989989888,$ $x_{11} = 98989989888,$ $x_{12} = 898989989888,$ and $x_{13} = 8898989989888,$ where the numbers $x_2,$ $x_3,$ $x_5,$ $x_8,$ $x_{10},$ $x_{12},$ and x_{13} were formed as in Case 1, and the numbers $x_4,$ $x_6,$ $x_7,$ $x_9,$ and $x_{11},$ were formed as in Case 2.

Editor's comments: We congratulate Mr. Nir for winning the Brilliancy Award from the Bay Area Mathematical Olympiad for a very elegant solution. The BAMO is a regional mathematical competition in the San Francisco area.

Solution 3 by Zhihao Liu (9/IL): Answer: 8898989989888

N is divisible by 2^{13}. Therefore for every integral value of k from 1 to 13, the last k digits of N are divisible by 2^k. Working from right to left, the unit, tens, and hundreds digits must all be 8, since 888 is divisible by 2^3.

Also, if a k-digit number is divisible by 2^k, then when it is divided by 2^{k+1}, the remainder is either 0 or 2^k. Using this fact, we can deduce the other digits of N. Since $888 \equiv 8 \equiv 9000 \bmod 16$, thus $9000 + 888 \equiv 8 + 8$, and 9888 is a multiple of 16. Similarly, $9888 \equiv 0 \bmod 32$, hence 32 divides 89888. Performing this algorithm eight more times:

$$89888 \equiv 32 \bmod 64$$

$$\Rightarrow 989888 \equiv 64 \bmod 128$$
$$\Rightarrow 9989888 \equiv 0 \bmod 256$$
$$\Rightarrow 89989888 \equiv 256 \bmod 512$$
$$\Rightarrow 989989888 \equiv 0 \bmod 1024$$
$$\Rightarrow 8989989888 \equiv 1024 \bmod 2048$$
$$\Rightarrow 98989989888 \equiv 0 \bmod 2048$$
$$\Rightarrow 898989989888 \equiv 0 \bmod 4096$$
$$\Rightarrow 8898989989888$$

Indeed, 8898989989888 is divisible by 2^{13} as it can be expressed as 8192x1086302489. Therefore, **N = 8898989989888**.

Editor's comments on modular arithmetic: This may be a good opportunity to briefly discuss modular arithmetic as used in Mr. Liu's solution above. Instead of the standard clock with twelve digits, *1, 2, 3, ..., 12*, consider a clock that has *p* digits, *0, 1, 2, ..., p-1*. As a specific example, consider a clock that has *p* = seven digits, *0, 1, 2, 3, 4, 5, 6*. With this clock we **add** as follows: *4 + 5* corresponds to the time starting at four o'clock and advancing five hours to nine o'clock, which on this clock is *2* o'clock. Thus, on this clock *4 + 5 = 2*. We write this $4 + 5 \equiv 2$, or to make sure everyone knows we are doing clock arithmetic on a clock with *7* digits, we might write $4 + 5 \equiv 2 \bmod 7$. We read this expression as *four plus five is congruent to 2 modulo seven*. Similarly, $6 + 3 \equiv 2$, $4 + 3 \equiv 0$, etc. Notice in each case, we only care what the remainder is after division by seven; in the example *4 + 5*, notice *4 + 5 = 9*, and *9* divided by *7* is *1* with a *remainder* of *2*, so $4 + 5 \equiv 2 \bmod 7$. We don't seem to care what the quotient is, we only care about the remainder after division by seven. We can also **subtract** with clock arithmetic: *4 - 5* corresponds to starting at four o'clock and backing up five hours, which on this clock is *6* o'clock. Thus, $4 - 5 \equiv 6$. Observe that $4 - 4 \equiv 4 + 3 \equiv 0$, so we may think of *3* as *-4*; that is the additive inverse of *4* is *3* [i.e. three is the number we must add to four to get zero]. Knowing how to add allows us to **multiply**: notice that 3×5 corresponds to adding 5 three times, and we already know how to add, so $3 \times 5 = 5 + 5 + 5 \equiv 1$. With this idea we can now multiply in this clock arithmetic system. Other examples of multiplication are $6 \times 3 \equiv 4$, $4 \times 3 \equiv 5$, etc. So now we can add, subtract, and multiply on this clock. **Division** is more difficult. If our clock has a prime number of digits, then we can always divide by nonzero numbers. But if our clock has *p* digits where *p* is not prime, we may not always be able to divide by every nonzero number. In our example with *p = 7* observe that $3 \times 5 = 5 + 5 + 5 \equiv 1$, so five is the multiplicative inverse of three [i.e. five is the number we must multiply by three to get one]. We write $\frac{1}{3} \equiv 5 \bmod 7$. Similarly, $\frac{1}{2} \equiv 4 \bmod 7$, $\frac{1}{4} \equiv 2 \bmod 7$, etc. Just to make sure you understand, before reading ahead, find $\frac{1}{6}$; that is, find the number you must multiply by *6* to get *1* modulo *7*.

When *p* is a prime, Euclid's algorithm helps us find multiplicative inverses. When *p* is *7* it is easy to do this just by guessing and trying. But if *p* is large, say *p = 163* or an even larger prime number, it is more difficult to guess and check. With *p = 163* we use the Euclidean algorithm as in the following example. Suppose we have the number 25 and we wish to find the multiplicative inverse of *25* modulo *163*. That is, we wish to find an integer *k* such that $25 \times k \equiv 1 \bmod 163$. The Euclidean algorithm gives us the greatest common divisor of *25* and *163*. Since *163* is prime and 0<25<163, we know this will be one. The process of the Euclidean algorithm gives us more information, which solves our problem. This process is sometimes called the extended Euclidean algorithm:

Divide 25 into 163 to get quotient 6 and remainder 13. $13 = 163 - 6 \times 25$.

Divide 13 into 25 to get quotient 1 and remainder 12. $12 = 25 - 1 \times 13$.

Divide 13 by 12 to get quotient 1 and remainder 1. $1 = 13 - 1 \times 12$.

Now back up through this process.

$$1 = 13 - 1 \times 12$$

$$1 = (1 \times 13) - 1 \times (25 - 1 \times 13) = (2 \times 13) - (1 \times 25)$$

$$1 = 2 \times (163 - 6 \times 25) - (1 \times 25)$$

$$1 = (2 \times 163) - (13 \times 25)$$

Thus, the greatest common divisor of 25 and 163 is one, and can be written as in the last equation above. Since I only care about the remainder after division by 163, and the term 2×163 will contribute zero to this remainder, I can write

$$1 \equiv -13 \times 25 \bmod 163$$

Since $-13 \equiv 150 \bmod 163$, we rewrite this as

$$1 \equiv 150 \times 25 \text{mo } 163.$$

So the multiplicative inverse of *25* modulo *163* is *150*. With clock arithmetic on a clock with *p = 163*, division by *25* is equivalent to multiplication by *150*.

Again, when p is prime we are guaranteed that each nonzero digit will have a multiplicative inverse, so we can divide by all nonzero digits. When *p* is not a prime we can still add, subtract, and multiply modulo *p*, but we cannot always divide by nonzero numbers because on these clocks there are nonzero divisors of zero.

$\frac{1}{6} \equiv 6 \bmod 7$ is the answer to the exercise posed above. Demonstrate it using Euclid's Algorithm.

Editor's comments: Thanks go to Professor George Berzsenyi, the creator of the USAMTS and our continuing supporter, for posing this interesting problem. Round 4 completes the tenth year of the USAMTS.

2/4/10. For a nonzero integer *i*, the exponent of 2 in the prime factorization of *i* is called $ord_2(i)$. For example, $ord_2(9) = 0$ since 9 is odd, and $ord_2(28) = 2$ since $28 = 2^2 \times 7$. The numbers $3^n - 1$ for $n = 1, 2, 3, \ldots$ are all even, so $ord_2(3^n - 1) > 0$ for $n > 0$.

a) For which positive integers *n* is $ord_2(3^n - 1) = 1$?

b) For which positive integers *n* is $ord_2(3^n - 1) = 2$?

c) For which positive integers *n* is $ord_2(3^n - 1) = 3$?

Prove your answers.

Solution 1 by Aaron Marcus (10/TX): (a.) $3^n - 1$ can be factored into $(3-1)(1+3+3^2+\ldots+3^{n-1})$. Since the second factor consists of only odd elements, the sum will be odd if and only if there are an odd number of elements. Since there are *(n-1)+1=n* elements (exponents *0* through *n-1*), $ord_2(3^n - 1) = 1$ when *n* is odd, or when $n = 2k + 1$.

(b.) Consider when n is even. The terms in the second factor can be paired. Thus,

$$3^n - 1 = (3-1)(1+3+3^2+\ldots+3^{n-1})$$
$$= 2((1+3\cdot 1)+(3^2+3\cdot 3^2)+\ldots(3^{n-2}+3\cdot 3^{n-2}))$$
$$= 2((4\cdot 1)+(4\cdot 3^2)+\ldots(4\cdot 3^{n-2}))$$
$$= 8(1+3^2+\ldots 3^{n-2})$$

Since n can only be even or odd, there is no case where $ord_2(3^n - 1) = 2$ (it must be either 1 or greater than or equal to 3).

(c.) From above, for $ord_2(3^n - 1) = 3$, $(1+3^2+\ldots+3^{n-2})$ must be odd.

$1+3^2+\ldots+3^{n-2} = 1+9+9^2+\ldots+9^{\frac{n}{2}-1}$. Since all terms are odd, this can only be odd if there are an odd number of terms. There are $\frac{n}{2}$ terms in the factor, so $ord_2(3^n - 1) = 3$ when $\frac{n}{2}$ is odd, or when $n = 4k + 2$.

Solution 2 by Reid W. Barton (10/MA): Use $\Leftrightarrow$ to represent *if and only if*. Note that

$$(\mathrm{ord}_2(n) = k) \Leftrightarrow 2^k \text{divides n and } 2^{k+1} \text{ does not divide n } \Leftrightarrow n \equiv 2^k (\mathrm{mod} 2^{k+1}) \quad .$$

(a.) Answer: *n* odd.

We have $\mathrm{ord}_2(3^n - 1) = 1 \Leftrightarrow 3^n - 1 \equiv 2 (\mathrm{mod}\ 4) \Leftrightarrow 3^n \equiv 3 (\mathrm{mod}\ 4)$. Now $3^1 \equiv 3 (\mathrm{mod}\ 4)$ and $3^2 \equiv 1 (\mathrm{mod}\ 4)$, so $3^n \equiv 3 (\mathrm{mod}\ 4)$ if and only if *n* is odd.

(b.) Answer: no *n*.

Since $\mathrm{ord}_2(3^n - 1) = 2 \Leftrightarrow 3^n - 1 \equiv 4 (\mathrm{mod}\ 8) \Leftrightarrow 3^n \equiv 5 (\mathrm{mod}\ 8)$, and $3^1 \equiv 3 (\mathrm{mod}\ 8)$, $3^2 \equiv 1 (\mathrm{mod}\ 8)$, there is no *n* for which $3^n \equiv 5 (\mathrm{mod}\ 8)$, and therefore no *n* for which $ord_2(3^n - 1) = 2$.

(c.) Answer: $n \equiv 2 (\mathrm{mod}\ 4)$.

in this case, $\mathrm{ord}_2(3^n - 1) = 3 \Leftrightarrow 3^n - 1 \equiv 8 (\mathrm{mod}\ 16) \Leftrightarrow 3^n \equiv 9 (\mathrm{mod}\ 16)$, and

$3^1 \equiv 3 (\mathrm{mod}\ 16)$, $3^2 \equiv 9 (\mathrm{mod}\ 16)$, $3^3 \equiv 11 (\mathrm{mod}\ 16)$, and $3^4 \equiv 1 (\mathrm{mod}\ 16)$, so $3^n \equiv 9 (\mathrm{mod}\ 8)$ if and only if $n \equiv 2 (\mathrm{mod}\ 4)$.

Editor's comment: This problem was posed by Dr. Erin Schram of the National Security Agency. Dr. Schram contributed one problem for each round of Year 10. His involvement is greatly appreciated.

3/4/10. Let f be a polynomial of degree 98, such that $f(k) = \frac{1}{k}$ for $k = 1, 2, 3, ..., 99$. Determine $f(100)$.

Solution 1 by Wei-Han Liu (8/TN): Consider the expression $f(x) - \frac{1}{x}$. This expression has roots at $1, 2, 3, 4, ..., 99$. However, this is not a polynomial. Next consider the function

$$g(x) = xf(x) - 1. \qquad \text{(a)}$$

Because $f(x)$ is a polynomial of degree 98, $g(x)$ is a polynomial of degree 99. It has the roots $1, 2, 3, 4, ..., 99$. Therefore it must be of the form

$$g(x) = c(x-1)(x-2)(x-3)\dots(x-99) \qquad \text{(b)}$$

where c is a constant. To determine c we find that $g(0) = -1$ from equation (a). So we put 0 in for x in equation (b) and get $g(0) = -c(99!) = -1$, so $c = 1/(99!)$ and

$$g(x) = \frac{(x-1)(x-2)(x-3)\dots(x-99)}{99!}. \qquad \text{(c)}$$

We want to find $f(100)$, so we plug 100 into both equation (a) and (c) and get

$$100(f(100)) - 1 = \frac{(100-1)(100-2)(100-3)\dots(100-99)}{99!}$$

$$100(f(100)) - 1 = 1$$

$$100(f(100)) = 2$$

$$f(100) = \frac{1}{50}$$

Therefore, $f(100) = \frac{1}{50}$.

Solution 2 by Daniel Moraseski (11/FL): Generalize by replacing 99 with n. Since $f(k)k = 1$ for all integers k between 1 and n,

$$f(x)x - 1 = A(x-1)(x-2)\dots(x-n),$$

where A is a constant, is also true for these values of k. Now plug in $x = 0$ and get

$$-1 = An!(-1)^n$$

$$A = \frac{(-1)^n}{n!}.$$

If we set A equal to this, the above equation is also true at $x = 0$. Since it is of degree n and is valid at $n+1$ values, it is true for all x. Now we find $f(n+1)$.

$$f(n+1)(n+1) - 1 = \frac{(-1)^{n-1}}{n!}n! = (-1)^{n-1}$$

$$f(n+1) = \frac{(-1)^{n-1} + 1}{n+1}$$

$f(n+1) = 0$ for n even. $f(n+1) = \frac{2}{n+1}$ for n odd.

Applied to our specific problem, $n = 99$, so $\boldsymbol{f(100) = 1/50.}$

4/4/10. Let A consist of 16 elements of the set $\{1, 2, 3, \ldots, 106\}$, so that no two elements of A differ by 6, 9, 12, 15, 18, or 21. Prove that two elements of A must differ by 3.

Solution 1 by Kasia Kobeszko (12/LA): (Proof by contradiction) Since all the differences forbidden within *A* (including the difference in question, *3*) are multiples of three, the set *A* can be separated into numbers coming from three disjoint sources: numbers in the form *3n*, in the form *3n+1*, and in the form *3n+2*, for integer *n*'s. The largest of these sources has *36* elements.

Let us assume that no two members of *A* differ by *3*.

This, combined with the stated difference restrictions, means that two members of *A* from any one source must differ by at least 24 to both be eligible for membership in *A*, or when counting by threes, they must be at least eight apart. That means that the most members that can come from one source is $\lceil(\text{size of source})/8\rceil \le \lceil 36/8 \rceil = 5$, where $\lceil w \rceil$ is the ceiling function indicating the smallest integer $\ge w$.

With at most *5* elements from each of three sources, *A* can have at most *15* elements. This contradicts the definition of *A* as a 16-element set, and disproves our assumption.

Solution 2 by Emily Kendall (10/IN): Assume temporarily that no two members of *A* differ by three. Consider all members of *A* congruent to *c* (mod *3*), where *c = 0, 1, or 2*. Because these members all differ by multiples of three, they must differ by at least 24.

106 can be divided into at most 4 intervals of 24. Since any two consecutive members of *A* congruent to *c* (mod *3*) must be separated by one of these intervals, there can be at most *5* such members.

Therefore, assuming that no two members differ by three, there are at most five members of *A* congruent to *1* (mod *3*), five congruent to *2* (mod *3*), and five congruent to *0* (mod *3*), for a total of *15*. So in order to have *16* elements in set *A*, two of them must differ by three.

Solution 3 by Vikki Kowalski (11/AR): What follows is an attempt to construct a set of 16 elements that meet all the required conditions of set *A and* have no two elements which differ by 3.

That would mean that for each new element added to *A*, 14 other potential elements are made invalid for consideration to become part of the set *A*. The maximum number of times that a single element could be counted ("overlap" between the set that one member of *A* makes invalid and the set that another member of *A* makes invalid) is twice.

This is because in order for an element to be removed from consideration twice, it must be *3n* (Let us define *n* and all integers n_i as integers such that $-8 < n < 8$) greater than a member of *A* and *3n* units less than a member of *A*. If an element was removed from consideration three times, this would mean that there must have been an element chosen for set *A* that was $3n_1$ from it and one $3n_2$ from it in the same direction (since there are only 2 directions and 3 elements, pigeonhole principle). However, this must mean that they are $3(n_1 - n_2)$ units apart, and $-8 < n_1 - n_2 < 8$, so they could not actually both be members of the set *A*.

The only problem that remains is the elements near the upper and lower limits of the set. If items for set *A* are selected sufficiently close to the first element in the set, they will each eliminate only *7* elements from consideration for set *A*. Without loss of generality, we may choose the elements near the lower limit of the set (*1, 2,* and *3*). Selecting these three will cause a total of only *8* elements to be eliminated from consideration (for each one selected). Following these, *3* elements may be selected at *24*-unit intervals (because these three, and each three following, eliminate a total of *24* elements). The maximum number which can be chosen for set *A* using this process before all possible elements have been either selected or eliminated from consideration is *15* (this may be repeated five times, terminating with *97*, *98*, and *99*).

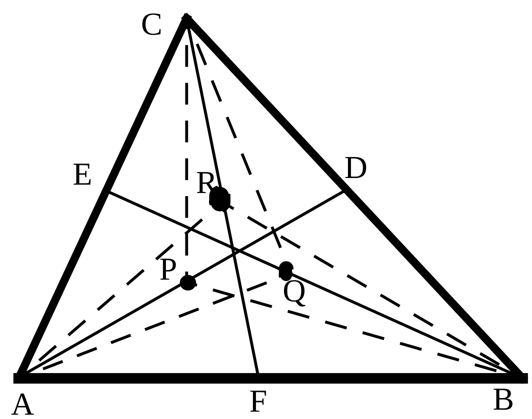
The only elements which are left to be selected that satisfy the initial constraints (but not the extra one applied originally) will be the ones which have a difference of *3* with one of the elements selected for *A*.

Therefore, two elements of *A* must differ by *3*.

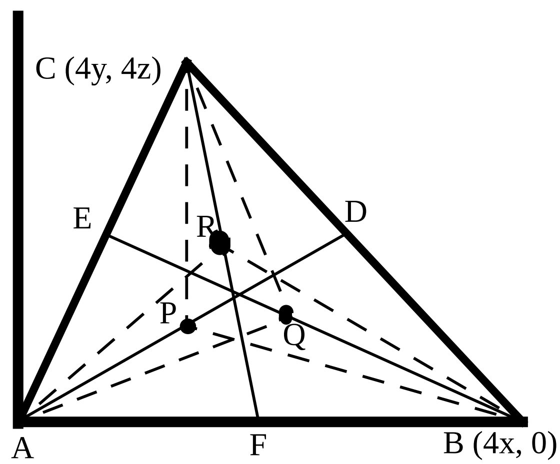
5/4/10. In ΔABC, let *D, E,* and *F* be the midpoints of the sides of the triangle, and let *P, Q*, and *R* be the midpoints of the corresponding medians, $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$, respectively, as shown in the figure at the right. Prove that the value of

$$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2}$$

does not depend on the shape of ΔABC and find that value.



Solution 1 by Luke Gustafson (10/MN): ΔABC can be placed on a coordinate system as shown on the right, with *A* on the origin, *B* at *(4x, 0)*, and *C* at *(4y, 4z)*. By the midpoint theorem, $D = (2x+2y, 2z)$, $E = (2y, 2z)$, $F = (2x, 0)$, $P = (x+y, z)$, $Q = (2x + y, z)$, and $R = (x+2y, 2z)$.



Using the distance formula,

$AQ^2 = (2x+y)^2+z^2 = 4x^2+4xy+y^2+z^2$

$AR^2 = (x+2y)^2+4z^2 = x^2+4xy+4y^2+4z^2$

$BP^2=(3x-y)^2+z^2= 9x^2-6xy+y^2+z^2$

$BR^2=(3x-2y)^2+4z^2= 9x^2-12xy+4y^2+4z^2$

$CP^2=(x-3y)^2+9z^2= x^2-6xy+9y^2+9z^2$

$CQ^2=(2x-3y)^2+9z^2= 4x^2-12xy+9y^2+9z^2$.

Adding these up gives $28(x^2-xy+y^2+z^2)$.

Furthermore,

$AB^2=16x^2$

$BC^2=16x^2-32xy+16y^2+16z^2$

$CA^2=16y^2+16z^2$

Adding these up gives $32(x^2-xy+y^2+z^2)$.

That makes $$\frac{AQ^2 + AR^2 + BP^2 + BR^2 + CP^2 + CQ^2}{AB^2 + BC^2 + CA^2} = \frac{28(x^2 - xy + y^2 + z^2)}{32(x^2 - xy + y^2 + z^2)} = \frac{7}{8}$$

Since this construction used an arbitrary triangle, this value is independent of the shape of ΔABC.