USA Mathematical Talent Search
Round 2 Grading Rubric
Year 36 — Academic Year 2024–2025
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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.
2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.
3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem.
4. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. If a student uses WolframAlpha, they must show their inputs and outputs. Merely citing the name of a software package is not sufficient justification.
5. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/2/36:

Award **5 points** for the correct configuration of entries. No justification is required. Withhold **1 point** for each entry that is incorrect.

Problem 2/2/36:

1 point: Student shows that the central number is m, where $3m$ is the magic sum.

1 point: Student uses a reasonable set of variables (such as the three variables m, p, and q in the official solution) to express the nine entries in the square. We did not award any credit for assigning nine variables to represent the nine entries, since this doesn't simplify the problem. That said, if such solutions were ultimately correct, we awarded a total score of **5 points**.

1 point: Student finds with explanation the solution with $m - p = 1$ and $q = 0$.

1 point: Student finds with explanation the solution with $m - p = 2$, $q = 1$, $p = 1$, and $m = 3$.

1 point: Student finds with explanation the solution with $m-p=2$, $q=1$, $p=2$, and $m=4$.

Note: Award a maximum of **4 points** if the student doesn't explain why no other solutions are possible, such as not explaining that if $m-p=2$, $q=1$, and $p \geq 3$, then all nine numbers are distinct.

Note: Award a maximum of **4 points** if the student includes additional solutions that are rotations or reflections, or in which all 9 entries are distinct.

Note: Award a total score of **1 point** for the correct numerical answer of 3 if there is no explanation. Award a total score of **2 points** if the student shows all three solutions, but doesn't provide additional explanation, or a total score of **1 point** if the student includes some but not all of the three solutions, or if the student has some incorrect solutions (or extra solutions such as rotations or reflections presented as additional solutions) alongside correct solutions.

Note: Students who write a computer program can receive full credit if they find all three solutions (and the program shows that there are no others), and if the student includes their code with sufficient supplementary explanation guiding the reader through the code. If there is an error in the computer program, it doesn't search an exhaustive set of configurations, or it includes extra solutions such as rotations or reflections, award partial credit as appropriate for significant constructive progress.

Problem 3/2/36:

Note: As with many geometry problems, expect a variety of solution methods. Any correct and complete solution should get **5 points**, with partial credit awarded as appropriate for solutions with errors or incomplete solutions, including for any of the specific items mentioned in the rubric.

Note: Award a maximum of **4 points** if the student only proves one direction of the if and only if statement. If the student's reasoning is reversible, but there is no discussion of reversibility, a score of **4 points** is appropriate. If the student proves the easier "if $\triangle EPQ$ is an equilateral triangle, then $\overline{AB} \perp \overline{CE}$" direction through a non-reversible argument, a score of **2 points** is appropriate.

Note: Do not deduct points due to configuration issues.

1 point: Student includes a helpful diagram.

1 point: Student uses angle chasing to obtain at least one set of meaningful results, such as finding the interior angle measures of triangle BPD.

1 point: Student notices that $\overline{AB} \perp \overline{CE}$ if and only if E lies on the perpendicular bisector of $\overline{AB}$, and uses this to obtain a significant result.

1 point: Student recognizes that $\angle AED = 2\angle ABD$ if and only if P lies on the circumcircle of triangle ADE.

1 point: Student shows that P lies on the circumcircle of triangle ADE if and only if triangle EPQ is equilateral.

Problem 4/2/36:

1 point: Student analyzes the expected contributions of both x_1 and x_2 to the expected value. Analyzing x_1 is **not** sufficient to get partial credit.

2 points: Student recognizes that for $i \geq 3$, each x_i contributes an expected value of

$$\frac{x_i}{2^i}\sum_{k=0}^{i-1}(-1)^k(k+1)\binom{i-1}{k}$$

and provides reasonable explanation for how they came up with the summation expression. Award **1 point** of partial credit if the summation expression is not well explained, for significant constructive progress towards this result, such as a detailed and correct analysis of x_3 combined with some attempt to generalize, or for a summation expression that is almost correct, but has a small error.

1 point: Student splits the summation into

$$\frac{x_i}{2^i}\left(\sum_{k=0}^{i-1}(-1)^k\binom{i-1}{k} + \sum_{k=0}^{i-1}(-1)^k k\binom{i-1}{k}\right).$$

and notes that the first summation is 0 by the binomial theorem.

1 point: Student shows that the second summation is 0.

Note: Award a total score of **1 point** if the student obtains the correct answer $\frac{x_1}{2} - \frac{x_2}{4}$, but does not provide any explanation.

Note: If the student does not fully simplify the summations, award partial credit as described above.

Note: Students often received a total score of **2 points** if they gave a non-rigorous explanation for why the x_i have an expected contribution of 0 for all $i \geq 3$; however, the overall score varied depending on the overall strengths and weaknesses of the solution.

Note: It is also possible to prove that the x_i have an expected contribution of 0 for all $i \geq 3$ using power series/generating functions and taking the derivative. If correct and well explained (and x_1 and x_2 were analyzed separately), these solutions received **5 points**.

Problem 5/2/36:

Official solution:

1 point: Student recognizes that if $a = \sqrt[3]{5}+\sqrt[3]{25}$, then a is a root of $P_1(x) = x^3-15x-30$.

2 points: Student recognizes that it is sufficient to show that there does not exist a polynomial $r(x)$ with integer coefficients of degree at most 2 such that $r(a) = 2\sqrt[3]{5} + 3\sqrt[3]{25}$. Award **1 point** of partial credit for significant constructive progress towards this observation.

1 point: Student shows that we cannot have $\deg r = 1$.

1 point: Student shows that we cannot have $\deg r = 2$. This case encompasses the $\deg r = 1$ case so a student who rigorously analyzes the $\deg r = 2$ case with no mention of the $\deg r = 1$ case can get full credit.

Note: The statement that $p\sqrt[3]{5} + q\sqrt[3]{5}$ is irrational whenever p and q are not both equal to 0, or equivalently that 1, $\sqrt[3]{5}$, and $\sqrt[3]{25}$ are linearly independent over $\mathbb{Q}$, is considered a well-known result that students can use without proof. If a student proves this result as part of an incomplete solution, award **1 point** of partial credit.

Alternate solution:

1 point: Student claims that $(\sqrt[3]{5} + \sqrt[3]{25})^n = A + B\sqrt[3]{5} + C\sqrt[3]{25}$ where $B \equiv C \pmod 4$. The student does not need to prove this claim to get this point.

1 point: Student connects the above claim to the goal of the problem by noticing that the coefficient of $\sqrt[3]{5}$, which is 2, and the coefficient of $\sqrt[3]{25}$, which is 3, are not equivalent mod 4.

3 points: Student provides a correct inductive or other proof of the above claim. If the student has the right ideas, but there is an error in the algebra, award **1 point** of partial credit. It is not required to show that two numbers $A + B\sqrt[3]{5} + C\sqrt[3]{25}$ for integers A, B, C are only equal when they have the same A, B, C, or that $p\sqrt[3]{5} + q\sqrt[3]{5}$ is irrational whenever p and q are both not equal to 0, but proving either of these results could justify **2 points** of partial credit depending on the student's overall progress towards the proof.

Note: A similar result can be proven by recognizing that in the expansion of $(\sqrt[3]{5}+\sqrt[3]{25})^n$, the terms for k and $n - k$ have equal binomial coefficients. Correct and complete solutions using this method should receive **5 points**. Withhold **1 point** if the $k = n/2$ case is missing, and withhold **2 points** if the binomial coefficients are worked out, but there is no proof of matching parity.