USA Mathematical Talent Search

Round 3 Grading Rubric

Year 35 — Academic Year 2023–2024

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.
2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.
3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem.
4. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. If a student uses WolframAlpha, they must show their inputs and outputs. Merely citing the name of a software package is not sufficient justification.
5. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/3/35:

Award **5 points** for the correct configuration of entries. No justification is required. Withhold **1 point** for each entry that is missing or incorrect.

Problem 2/3/35:

Solutions similar to the official solution:

1 point: Student recognizes that a must be 3 or 4 (assuming WLOG $a \le b \le c$).

2 points: Student successfully analyzes the $a = 3$ case. Award **1 point** of partial credit if the student makes significant constructive progress, such as finding that $5 \le b \le 8$ and finding at least one of the solutions when $a = 3$.

2 points: Student successfully analyzes the $a = 4$ case. Award **1 point** of partial credit if the student makes significant constructive progress, such as finding that $4 \le b \le 5$ and finding at least one of the solutions when $a = 4$.

Note: If the student successfully finds the bounds on b for both cases, but does not find any of the corresponding values of c, award **1 point** out of the final **4 points**.

Note: If a student does not find the number of cubes corresponding to a set of values for a, b, and c, do **not** award **5 points**.

Note: Don't be overly punitive for mistakes that carry through the solution. For example, if a student incorrectly determines the possible values of a, but does a correct analysis of the values of b and c for their values of a I wouldn't give the student the first point, but would consider awarding at least **1 point** out of the final **4 points**.

Note: Award a total score of **1 point** if the student finds all four solutions with no explanation. Do not award any credit if the student finds some (but not all) solutions with no explanation.

Computer solutions:

Note: If a student uses a computer program as part of their solution, make sure they have reduced the problem to checking a finite number of cases. If the student doesn't find bounds for a, b, c, award at most **1 point**. Relatedly, if a student's program specifies a particular bound such as $a, b, c \leq 100$, they need to show that a, b, c satisfy this bound to get full credit.

Note: Award a total score of **3 points** for an otherwise correct solution that depends on floating-point equality. Because computers do not work to infinite precision, and store decimal numbers in binary, $5 \cdot 0.2$ may be 0.99999999 rather than 1 on some computers. It is acceptable to compute $c = x/y$ and then check whether c==int(c), as x/y will be exact if it is an integer.

Solutions based on at least one of a, b, c being a multiple of 5:

Note: Do not award credit for proving that at least one of a, b, c must be a multiple of 5 since this is trivial.

1 point: Student proves that if a is a multiple of 5, then b or c must be 3 or 4 (as in the official solution).

1 point: Student proves that $a \leq 20$.

1 point: Student shows how the factorization works given a and at least one value of b, either finding a solution or proving that there are none.

Note: Award a total score of **3 points** if the student covers multiple cases, or covers a case with multiple solutions (e.g., $b = 3$ for all values of a divisible by 5) but is missing cases. This includes finding all four solutions but not excluding $a > 20$.

4 points: Student does a correct case analysis, but misses a solution because of an arithmetic error.

Problem 3/3/35:

1 point: Student comes up with a useful framework for analyzing the game, such as the geometrical interpretation on the first page of the official solution, and makes at least one significant observation about hot and cold positions (e.g., $(1, 0)$ and $(0, y)$ with $y > 0$ are hot).

2 points: Student recognizes that $(3, 0)$ is cold. Award **1 point** of partial credit for significant constructive progress towards this result, such as recognizing that $(1, 1)$ is cold and making at least one additional useful observation about hot and cold points.

2 points: Student continues the pattern to show that Lizzie has a winning strategy if and only if n is of the form $3k + 1$ or $3k + 2$. Award **1 point** for significant constructive progress towards achieving this generalization, such as the last figure in the official solution.

Note: If the student obtains the correct answer with no explanation, award a total score of **1 point**.

Note: Do not award any credit if the student misinterprets the problem to say that only one number can be reduced at at time. This misinterpretation violates the second type of move, which says that a player can "subtract one from any positive number of positive numbers on the board."

Note: Do not award any credit if the student incorrectly analyzes the game as if it were Nim.

Problem 4/3/35:

Note: Part (a) is worth **2 points** and part (b) is worth **3 points**. "Half-points" from part (a) and part (b) should be combined when determining the total score. So, if you would give the student **1.5 points** for part (a) and **0.5 points** for part (b), award a total score of **2 points**.

Part (a): Award **1 point** for a function that works, and award an additional **1 point** for the explanation of why the function works.

Part (b): Award **1 point** for any meaningful result, such as that f must be injective or that if A, B, and C are not collinear, then $f(A)$, $f(B)$, and $f(C)$ are not collinear. Award **2 points** for a valid counterexample such as a regular pentagon with proof. Award **1 point** of partial credit for significant constructive progress towards this result, such as analyzing the possibilities that the convex hull is a quadrilateral or pentagon.

Note: If no or minimal explanation is provided, don't award any credit for the correct yes/no answers.

Problem 5/3/35:

Note: Students need to do two things here. First, they need to show that if a point P satisfies the conditions in the problem, then it lies on the ellipsoid $x^2 + y^2 + 2z^2 = 1$. This is worth **3 points**. Second, they need to show that any point on the ellipsoid satisfies the conditions in the problem. This is worth **2 points**. The former was much more common than the latter, but if the student only does the latter, they should receive **3 points**.

Note: Given the complexity of the problem and that it is Round 3 Problem 5, the threshold for partial credit is higher than in other geometry problems. Among other things, we are not giving credit just for including a diagram that incorporates the given information. That said, the student solution should include diagrams as needed, and you should take off points accordingly if the student doesn't include sufficient diagrams to make their solution reasonably easy to follow.

1 point: Student shows that the projection of P onto the xy-plane is the orthocenter of $\triangle ABC$.

1 point: Student uses coordinate geometry to get meaningful expressions for $PA^2 + PB^2 = AB^2$.

1 point: Student uses algebra to show that P must be on the ellipsoid $x^2 + y^2 + 2z^2 = 1$.

2 points: Student shows that any point on the ellipsoid satisfies the conditions in the problem. Award **1 point** of partial credit for very significant constructive progress towards this result, such as showing that if H is the projection of a point on the ellipsoid onto plane ABC, then H is the orthocenter of $\triangle ABC$.

Note: We tended to be generous when evaluating student explanations for why any point on the ellipsoid satisfies the conditions in the problem, but students do need to address this in their solution.

Note: If the student obtains the correct answer with no supporting explanation, award **1 point**.