USA Mathematical Talent Search

Round 1 Grading Rubric

Year 35 — Academic Year 2023–2024

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.
2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.
3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem.
4. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. If a student uses WolframAlpha, they must show their inputs and outputs. Merely citing the name of a software package is not sufficient justification.
5. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/1/35:

Award **5 points** for the correct configuration of entries. No justification is required. Withhold **1 point** for each entry that is missing or incorrect.

Problem 2/1/35:

5 points: Student shows that the concatenated number must be divisible by 101. Two common methods were the official solution and proposing a divisibility test for 101. If the student proposes a divisibility test for 101, they need to explain why their divisibility test works to get full credit. If the student's divisibility test is valid, but not well-explained, award at most **4 points**. (We typically awarded **3 points** if the student made no attempt to prove the divisibility test.)

Some things that should earn points are as follows:

1 point: Student recognizes something useful about the structure of the concatenated number, such as

$$n = a_{100} \cdot 10{,}000^{100} + a_{99} \cdot 10{,}000^{99} + \cdots + a_1 \cdot 10{,}000^1 + a_0 \cdot 10{,}000^0.$$

1 point: Student recognizes that 10,000 is 1 more than a multiple of 101 (or that 9999 is a multiple of 101) and attempts to do something useful with this.

1 point: Student explains why the concatenated number isn't divisible by 3 and/or 11. Do not award this point if the student makes an error when performing these divisibility tests, such as incorrectly computing the sum of the digits.

Note: Do **not** award any points for the correct answer if there is no explanation. If the student's answer is correct and they make a good-faith effort to explain their reasoning, award at least **1 point**.

Problem 3/1/35:

1 point: Student proves that the minimum number of roof-friendly pairs is $n - 1$. It is sufficient to give an example, such as the arrangement $1, 2, 3, \ldots, n$.

4 points: Student proves that the maximum number of roof-friendly pairs is $2n - 3$. Award these points as follows:

1 point: Student claims that the maximum is achieved when the tallest building is at one end and the second-tallest building is at the other end.

2 points: Student explains why the maximum cannot be achieved if this condition is not satisfied. Award **1 point** of partial credit for significant constructive progress towards this result, such as noticing that there can be no roof-friendly pair where the two buildings are on opposite sides of the tallest building.

1 point: Student explains why $2n-3$ roof-friendly pairs can be achieved when the tallest building is at one end and the second-tallest building is at the other end.

Note: Award **1 point** total if the student states both the minimum and maximum number of roof-friendly pairs in terms of n with no explanation. Stating only one of these expressions without proof is insufficient to get any credit.

Note: Award a total score of **2 points** if the student has a correct construction of both the minimum and maximum, with a proof of the minimum but no proof of the maximum. A common issue was students using the "greedy algorithm" to obtain the arrangement $n, n-2, n-4, \ldots, 2, 1, 3, \ldots n-1$ (if n is even) and/or $n, n-2, n-4, \ldots, 1, 2, 4, \ldots n-1$ (if n is odd), and assuming that this construction maximizes the number of roof-friendly pairs. If a student did not explain why it is impossible to get more roof-friendly pairs, we typically gave a score of **2 points**.

Problem 4/1/35:

1 point: Student recognizes that the left sides of the given equations relate to Vieta's formulas.

1 point: Student uses their understanding of Vieta's formulas and the right sides of the given equations to recognize that (a, b, c) is a solution if and only if a, b, c are roots of the polynomial

$$P(z) = \frac{Q(z) + R(z)}{2}$$

where

$$Q(z) = (z - x)(z - x^2)(z - x^4)$$

and

$$R(z) = (z - y)(z - y^2)(z - y^4).$$

2 points: Student successfully analyzes the $x \neq y$ (with $y \neq x^2$) case (WLOG $y > x$ in the official solution) to show that P has three real roots. Award **1 point** for significant constructive progress, such as finding an interval for at least one of the roots.

1 point: Student successfully analyzes the additional cases, specifically $y = x^2$ and $x = y$. Do not deduct points for missing the trivial $x = y = 1$ case, and do not award credit if a student only covers the trivial case. More generally, do not award any points for analyzing some special cases if there is no use of Vieta's formulas.

Note: Some students attempted to prove that there are three real roots because the cubic discriminant is always nonnegative. There were no successful solutions using this approach. We typically awarded **1 point** if the student used Vieta's formulas or **0 points** otherwise.

Problem 5/1/35:

Note: Many students used calculators and/or WolframAlpha to help them deal with complicated trig expressions. If a student used WolframAlpha and clearly specified their inputs and outputs, the submission was eligible to receive **5 points** provided that everything was correct. We prefer to see the raw inputs in case a student messed up parentheses, etc. when trying to input a complicated expression into WolframAlpha, but gave credit even if the student showed their input using LaTeX, presumably to make it easier for us to read. If the student did not clearly specify the inputs and outputs, we did not award credit for the corresponding steps. It is acceptable to include screenshots of your WolframAlpha inputs and outputs.

It is important for students to understand both the benefits and limitations of technology. This means understanding that many calculators (e.g., Desmos) use approximations. For example, if I ask Desmos to compute $2^{-5000} - 2^{-5001}$, it will give an output of 0 even though this is not the exact value. Thus, we gave a maximum of **4 points** to students who said that they used "a calculator" without specifying which calculator they used, or if they said they used a calculator such as Desmos that does approximations. We also gave a maximum of **4 points** if students showed that the shaded and unshaded areas were equal through a finite number of decimal places. This is not a rigorous proof, since the next decimal place could conceivably be different. If a student used intermediate rounding throughout, we awarded at most **3 points**.

For the official solution, award points as follows:

1 point: Student splits the polygon into triangles and finds the three pairs of congruent triangles whose areas cancel.

2 points: Student relates the areas of the six red and blue triangles to the areas of triangles with side $\overline{A_7A_8}$. Award **1 point** for recognizing that $[A_7A_8B] = [A_7A_8C]$ and **1 point** for the other five triangles.

2 points: Student finishes showing that the total area of the red triangles is equal to the total area of the blue triangles. Award **1 point** for significant constructive progress towards this result, such as recognizing that $A_5CA_{10}E$ and $A_3EA_{12}A_1$ are rhombi.

Note: Some of these are big steps, so consider awarding credit as appropriate for significant constructive progress towards each of these steps. For example, if the student splits the polygon into triangles, finds one pair of congruent triangles whose areas cancel, and makes no other significant progress, still give the student **1 point**.

Award **5 points** for a successful trig bash. Award at least **1 point** if the student sets up a trig bash and makes meaningful constructive progress. Award additional partial credit as appropriate based on how close the student's trig bash is to a complete and correct solution. (There was lots of variability in the trig expressions students obtained, so we tended to rely on this general guideline.)