USA Mathematical Talent Search

Round 1 Grading Rubric

Year 34 — Academic Year 2022–2023

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/1/34:

Award **5 points** for the correct configuration of arcs. No justification is required. Withhold **1 point** for each arc that is missing or drawn incorrectly.

Problem 2/1/34:

1 point: Student finds that the answer is all odd n. No justification is required to earn this point.

1 point: Student recognizes that it is helpful to consider the coloring of a point and its antipode.

1 point: Student recognizes that going from a point to its antipode involves crossing all n great circles exactly once.

1 point: Student explains why if n is odd, a point and its antipode have different colorings, and thus why all good colorings must have equal black and white areas.

1 point: Student explains why if n is even, a point and its antipode have the same coloring, and thus why it is not necessarily true that all good colorings must have equal black and white areas.

Note: Award a total score of **3 points** for a correct proof that all odd n require a "balanced" coloring, with no attempt to prove that an "unbalanced" coloring is possible for all even n.

Note: Award **2 points** for a showing a good but unbalanced coloring for all even n, as long as it is clear why the coloring works. Award additional credit for observations that work for odd n.

Note: Award a total score of **2 points** for an argument based on projecting the great circles onto the equator. This does give an unbalanced coloring for even n, but a great circle that does not go through the poles does not project to a line on the equator. (It projects to an ellipse, and areas are not preserved.

Note: If a student successfully analyzes both the cases of $n = 1$ and $n = 2$, award **1 point** of partial credit.

Problem 3/1/34:

1 point: Student proposes a sufficient claim to prove by induction, and student proves the base case $n = 1$.

1 point: Student correctly describes the inductive hypothesis, and claims that we append 1 or 2 to the front of M to get N.

2 points: Student proves that it is possible to construct N from M in this way. Award **1 point** for significant constructive progress towards this result, such as arriving at the equation

$$N = d \cdot (1011)^{k-1} \cdot 2^{k-1} + m \cdot 2^{k-1} = \left(d \cdot (1011)^{k-1} + m\right) \cdot 2^{k-1}$$

or a comparable equation. Award an additional **1 point** for correctly analyzing this equation to find the appropriate choice of d.

1 point: Student proves the uniqueness of N.

Note: Some students successfully showed by induction that we get every residue mod 2^n exactly once, so we get a unique n-digit number that satisfies the conditions in the problem and is a multiple of 2^n. We graded these solutions as follows:

Award **1 point** for the claim that we get each residue exactly once.

Award **1 point** for the right setup and some non-trivial progress. For example, award a point to any solution that contains "Assume $A \equiv B$ and $A \neq B$" and also writes $A - B$ as a useful sum.

Award **2 points** for showing that if $A \equiv B$, then $A = B$. Award 1 point for any constructive progress that considers divisibility by powers of 2.

Award **1 point** for tying everything together (showing the other direction of the one-to-one correspondence, which is showing uniqueness instead of existence).

Problem 4/1/34:

1 point: Student obtains the correct answer of \$75. No justification is required to earn this point.

2 points: Student describes Winnie's optimal strategy, and explains how it guarantees that she wins at least \$75. Award **1 point** for describing the strategy, and **1 point** for the explanation.

2 points: Student describes Grogg's optimal strategy, and explains how it guarantees that he loses at most \$75. Award **1 point** for describing the strategy, and **1 point** for the explanation.

Note: If the student uses the greedy algorithm to get an answer of \$27, the submission should receive **1 point**.

Note: A lot of students came up with incorrect strategies for Winnie and Grogg. If the student's analysis of the incorrect strategies captures key ideas from the correct strategies, we considered awarding **1 point** at the discretion of the grader.

Problem 5/1/34:

Note: Part (a) is worth **1 point** and part (b) is worth **4 points**. If a student finds an incorrect function in part (a), award points for the student's work in part (b) if the student's analysis would have been significant constructive progress had the student found the correct function. Use the key points mentioned below as a guide.

Part (a):

1 point: Student finds that $g(x) = x^2 + 2x + 37$. Award this point even if the student doesn't prove irreducibility.

Part (b):

Note: Many students used casework, but the specific organization of the casework varied across students.

Award full credit for a complete and correct solution using casework or another method. Award **2 points** for a proof that omits the case in which n is a cube and thus misses $n = 27$ or omits the case in which n is any large power of a prime, but is otherwise correct. Award only **1 point** if other cases are missing, such as $n = p^2 = q$.

Note: If the student provides the answer of 27 without any explanation, award a total score of **1 point** for part (b).

Note: If the student only omits the case $n = 1$, do not deduct a point.

Note: If the student includes the sixish numbers in the set of pseudo-sixish numbers, do not deduct a point if the solution is otherwise correct.