USA Mathematical Talent Search

Round 2 Grading Rubric

Year 32 — Academic Year 2020–2021

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GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to earn **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/2/32:

Award **5 points** for the correct configuration of numbers. No justification is required. Withhold **1 point** for each incorrect entry.

Problem 2/2/32:

1 point: Student explains why another beast will temporarily hold all the candy if $n = 1$ and $n = 2$. The remaining **4 points** are for showing that this does not occur if $n > 2$.

1 point: Student makes a significant observation, similar to one of the lemmas in either official solution.

3 points: Student proves the lemma (**2 points**, award **1 point** of partial credit for significant constructive progress towards proving the lemma), and explains how the lemma allows us to complete the solution (**1 point**).

Note: Award partial credit as appropriate if the student makes additional significant constructive progress, but is unable to complete the solution. When determining how much partial credit to award, consider whether the student proved their key observations or stated them without proof.

Problem 3/2/32:

1 point: Student recognizes that Vieta's formulas are useful for solving this problem, and makes a reasonable attempt to apply Vieta's formulas to the problem.

2 points: Student successfully analyzes the $n = m$ case. Award **1 point** for recognizing that we can have $a = b$, which gives us $S(p+q) = 8$, and finding valid polynomials $p(x)$ and $q(x)$ that satisfy this condition. Award **1 point** for recognizing that we can have $\frac{7a-9b}{a-b} = 11$ (equivalently $b = 2a$), which gives us $S(p+q) = \frac{25}{3}$.

1 point: Student successfully analyzes the $n = m+1$ case, including finding valid polynomials $p(x)$ and $q(x)$ that satisfy $\frac{7a+b}{a} = 11$ (equivalently $b = 4a$), which gives us $S(p+q) = 3$.

1 point: Student successfully analyzes the $n = m-1$ case, including finding valid polynomials $p(x)$ and $q(x)$ that satisfy $\frac{-a-9b}{-b} = 11$ (equivalently $a = 2b$), which gives us $S(p+q) = 7$.

Note: Example polynomials were required for full credit in the subcase of $m = n$ in which $a = b$ because in this case $p(x) - q(x)$ is of lower degree and thus the sum of the roots is not determined by the first two coefficients. In the other cases, the attainability of $p(x)$ and $q(x)$ is implicit from the application of Vieta's formulas.

Note: If the student doesn't explain why $n = m$, $n = m+1$, and $n = m-1$ are the only possible cases, award at most **4 points**.

Problem 4/2/32:

Note: The rubric below applies to the first official solution, though as is usually the case with geometry problems, there was some variation in the student methods. Since several students did complex bashing, we have included some comments about those solutions as well. One of the graders was kind enough to write a complex bashing solution (modeled on several student solutions), which we have added as a second official solution. This solution is intended to give students a sense of what a valid complex bashing solution to this problem looks like, as well as what is a good level of detail for bashing solutions.

Note: The problem contains a figure, so it may not be essential for the student to include an updated figure of their own. But if a student adds a lot to the given figure, an updated figure may be necessary.

Note: Students do not need to prove the ratio lemma in their solutions. This counts as a well-known result that can be cited without proof.

1 point: Student sketches a viable solution strategy (e.g., recognizing that it is sufficient to show that $\frac{AX}{XT} = \frac{HY}{YC}$) and makes at least some meaningful constructive progress towards proving the desired result.

2 points: Student obtains

$$\frac{AX}{XT} = \frac{\cos\angle BAC}{\cos\angle CBA} \cdot \frac{\sin(\angle CBA - \angle ACB)}{\sin(\angle BAC + \angle ACB)}$$

or a reasonably equivalent result. To receive **1 point** of partial credit, it is sufficient to apply the ratio lemma to $\triangle ACT$ with point X on AT or to do a reasonable amount of constructive angle chasing.

2 points: Student obtains

$$\frac{HY}{YC} = \frac{\cos\angle BAC}{\cos\angle CBA} \cdot \frac{\sin(\angle CBA - \angle ACB)}{\sin(\angle BAC + \angle ACB)}$$

or a reasonably equivalent result. To receive **1 point** of partial credit, it is sufficient to apply the ratio lemma to $\triangle HAC$ with point Y on HC or to do a reasonable amount of constructive angle chasing.

Note: In the bashing solution, it is valid to assume WLOG that the vertices of the triangle all lie on the unit circle.

Note: In the next-to-last step of the bashing solution, it is not valid to start with the equation $\frac{b+2c}{a} = \overline{\left(\frac{b+2c}{a}\right)}$ and work towards showing that $2b\left(c^2 - a^2\right) = c\left(a^2 - b^2\right)$. Instead, the student should work in the order shown in the solution, or at least indicate that all of the steps are reversible. We deducted **1 point** if a student made this mistake.

Problem 5/1/32:

1 point: Student states and proves Lemma 1a.

1 point: Student states and proves Lemma 1b.

1 point: Student states and proves Lemma 2.

1 point: Student states and proves Lemma 3.

1 point: Student explains how the lemmas allow us to reach the conclusion that $a_N = a_{N+2^{2020}}$ for all sufficiently large N.

Note: Proving that the sequence is eventually periodic with period at most 2^{2020} is worth **1 point**.

Note: The general guideline of awarding (at least) **1 point** for any meaningful observation with proof applies to this problem.