USA Mathematical Talent Search

Round 1 Grading Rubric

Year 31 — Academic Year 2019–2020

www.usamts.org

GENERAL GUIDELINES

1. The grading rubric is designed to be simultaneously **specific** and **flexible**. For common solution methods, the rubric provides a specific allocation of points to ensure consistency across graders. Less common solution methods might not be captured closely by the rubric. For less common solution methods, consider the amount of constructive progress (including any specific intermediate results discussed in the rubric) and how far or close the student is to a complete solution when determining the score.

2. On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

3. Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

4. A student's justification needs to be rigorous and reasonably clear in order for the solution to merit **5 points**. If there is a meaningful gap in the student's argument or a key step is unclear, deduct points accordingly.

Problem 1/1/31:

Award **5 points** for the correct configuration of squares and dominoes. No justification is required. For grading purposes, an error is defined as a unit square in the grid that has either the wrong type of piece (square instead of domino or vice versa), or a domino in the incorrect orientation. Give **4 points** if there are 1-2 errors, **3 points** if there are 3-4 errors, **2 points** if there are 5-6 errors, **1 point** if there are 7-8 errors, and **0 points** if there are more than 8 errors.

Problem 2/1/31:

Note: The rubric is based on the official solution. Most student submissions contained elements of the official solution, but did not necessarily present the solution in exactly the same way. For example, many students did not formally propose a proof by contradiction, yet obtained similar intermediate results, which were useful for determining that $x = y = z$. As a reminder, any correct and complete solution should receive **5 points**, and the intermediate results in the rubric are a guide for how to award partial credit.

1 point: Student proposes a proof by contradiction (with a specific assumption for the sake of contradiction) or other valid strategy for solving the problem.

1 point: Student shows that if $x < y$, then $y > z$.

1 point: Student shows that if $x < y$, then $x < z$.

1 point: Student shows that if $y > z$, then $z < x$.

1 point: Student completes the proof by contradiction by pointing out that we can't simultaneously have both $x < z$ and $z > x$.

Note: Some students incorrectly assumed WLOG that $x < y < z$; the case of $x < z < y$ is not equivalent. These solutions typically received **3 points**.

Problem 3/1/31:

Note: Some students generated geometric solutions (similar to the official solution), and other students generated algebraic solutions (e.g., coordinate bashing). There is a separate rubric for each type of solution.

Geometric Solutions

1 point: Student creates a meaningful figure.

3 points: Student establishes $\frac{1}{4}$ as an upper bound for the area of $\triangle PIE$ with justification. These points should be awarded as follows: student makes a meaningful addition to the diagram (beyond the information given in the problem) and makes at least one useful observation (e.g., similar triangles) (**1 point**), student identifies another triangle (e.g., $\triangle OIE$) whose area is related to $\triangle PIE$ and explains the relationship between the areas (**1 point**), student finds the area of the other triangle and uses this information to determine the upper bound of $\frac{1}{4}$ for $[PIE]$ (**1 point**).

Award **1 point** of partial credit out of the above 3 points if the student proves a weaker, but meaningful upper bound. Showing that $[PIE] \leq \frac{1}{2}$ isn't sufficient to get any partial credit, but anything better than that is sufficient to get partial credit.

1 point: Student proves that $[PIE] = \frac{1}{4}$ is attainable.

Algebraic Solutions

1 point: Student creates a meaningful setup using coordinates.

3 points: Student uses their setup to determine the upper bound of $\frac{1}{4}$ for $[PIE]$. Award partial credit based on the amount of constructive progress, and how close or far the student is from this result.

1 point: Student proves that $[PIE] = \frac{1}{4}$ is attainable.

Note: For all submissions, if the absence of a figure made a solution very hard to follow, we only awarded credit for work the grader was able to verify with reasonable effort.

Problem 4/1/31:

1 point: Student recognizes that it will be helpful to find a quantity that is invariant under the processes of sharing and eating.

1 point: Student proposes a valid invariant.

1 point: Student shows that the quantity is invariant under the process of sharing.

1 point: Student shows that the quantity is invariant under the process of eating.

1 point: Student makes the critical connection to binary, and explains why there must be 8 mangos at the end.

Note: If the student shows at least one complete and correct scenario in which 8 mangos are left at the end, but makes no other significant constructive progress, award **1 point**.

Note: Some students incorrectly claimed that a mango cannot go all the way around the circle in a leftward direction. These solutions received at most **4 points**.

Note: Some students noticed that if person i shares and then person $i-1$ (the person directly to the left) eats, the resulting configuration of mangos is the same as if person i had eaten. These students typically tried to reduce the problem to the scenario in which people only eat. This approach doesn't work if there are more than 2011 instances of sharing. The most common score for this type of solution was **3 points**, but the score varied based on the overall rigor of the student's explanations, as well as whether the student made additional constructive progress.

Note: Some students only considered a limited range of combinations of sharing and eating. In these cases, the score depended on the combinations of sharing and eating that the student analyzed, as well as the extent to which the student succeeded at making meaningful generalizations.

Problem 5/1/31:

Note: As usual, there is some variation in student solution methods. In the following rubric, I've highlighted some key intermediate results that are present in a decent number of student solutions.

3 points: Student shows that $\frac{S_n}{n^3} \leq \frac{1}{2}$. Award appropriate partial credit for constructive progress towards this result. The intermediate results discussed below are not intended to be the only results worthy of partial credit; award equivalent partial credit for any equivalent progress.

To obtain **1 point** of partial credit, it is sufficient to notice that the sum

$$\sum_{a=0}^{n-1} r_n(a,b) = r_n(0,b) + r_n(1,b) + \cdots + r_n(n-1,b)$$

repeats itself every $\frac{n}{d}$ terms, where $d = \gcd(b,n)$.

To obtain **2 points** of partial credit, it is sufficient to determine that

$$\sum_{a=0}^{n-1} r_n(a,b) = \frac{n(n-d)}{2}.$$

2 points: Student shows that $\frac{S_n}{n^3} \geq \frac{1}{2} - \frac{1}{\sqrt{n}}$.

Award **1 point** of partial credit for significant constructive progress towards this result. The meaningful use of the Euler totient function is sufficient (but not the only way) to earn the point of partial credit.

Note: If a student proved both the upper and lower bounds when n is prime and stated that this is the case that was covered, we awarded **2 points**.

Note: Some students noticed that $r_n(a,b) + r_n(a,n-b)$ can equal either 0 or n. This is a valid approach. If a student did not recognize that in some cases $r_n(a,b) + r_n(a,n-b) = 0$, the solution received at most **2 points**.