USA Mathematical Talent Search

Round 3 Grading Rubric

Year 28 — Academic Year 2016–2017

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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

IMPORTANT NOTE: Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

Problem 1/3/28:

Award **5 points** for the correct configuration of numbers. No justification is required. Withhold **1 point** for each incorrect entry.

Problem 2/3/28:

1 point: Student correctly analyzes Fames's first statement: the day of Malmer's birthday is not 1. (This also tell us that the number of the month of Malmer's birthday is not 1.)

1 point: Student correctly analyzes Weven's first statement: the number of the month of Malmer's birthday is not 12. (This also tells us that the day of Malmer's birthday is not 12.)

2 points: Student recognizes that in each subsequent exchange, Fames's statement rules out the lowest remaining possible number for the day (and month) and Weven's statement rules out the highest possible remaining number for the month (and day).

1 point: Student concludes that Malmer's birthday is July 7. Award this point even if no justification is provided.

Problem 3/3/28:

Note: Part (a) is worth **1 point** and part (b) is worth **4 points**.

Part (a): Award **1 point** for a valid 5-city that has some entry that is at least 150. No justification is required. If any entries are incorrect, award **0 points**.

Part (b): Award the **4 points** for this part as follows:

1 point: Student recognizes that an n-city must have at least one 1 in every row and column.

1 point: Student recognizes that $a_k \le F_{k+2}$, where a_k is the k^{th} entry when the entries are arranged from least to greatest and F_m is the m^{th} Fibonacci number.

2 points: Student shows that $F_{2\binom{n}{2}+2} \le 3^{\binom{n}{2}}$. Some methods include the proof by induction in the official solution, Binet's formula, and the fact that the ratio of consecutive Fibonacci numbers approaches the golden ratio ϕ. (Award **1 point** out of these two points for significant constructive progress towards a correct proof of the key claim.) In order to get full credit, students using a ratio-based argument must also verify that the claim we are trying to prove is true for small values of n.

Problem 4/3/28:

5 points: Student provides a complete and correct solution.

Few students used a method similar to the official solution. A more common approach involved assuming that $M_A = (1, 0)$ and $M_B = (0, 0)$, where M_A and M_B are the centers of mass of sets A and B respectively (i.e., the two sets of points have different centers of mass). Students could then perform a rigorous analysis showing that when $x = (t, 0)$ for a very large value of t, we have

$$D(x, A_1) + D(x, A_2) + \cdots + D(x, A_n) < D(x, B_1) + D(x, B_2) + \cdots + D(x, B_n).$$

Students could also use limits to explore what happens to the values of $\sum D(x, A_i)$ and $\sum D(x, B_i)$ when $x = (r, 0)$ and $x = (-r, 0)$ as r approaches infinity. It is possible to show both that (1) the x-coordinate of M_A is greater than or equal the x-coordinate of M_B and (2) the x-coordinate of M_A is less than or equal the x-coordinate of M_B. So, the x-coordinates of M_A and M_B are equal. A similar argument needs to be made for the y-coordinates; we can let $x = (0, s)$ and $(0, -s)$ as s approaches infinity.

4 points: Student provides a nearly complete and correct solution, but is missing a non-trivial step. (If the missing step is especially important, award fewer points.)

3 points: Student sets up a mathematical argument using limits or a large value of t and makes significant constructive progress, but there are significant errors in the analysis.

2 points: Student sets up a mathematical argument using limits or a large value of t, but does not make significant constructive progress.

1 point: Student proves a useful result (e.g., a correct analysis of the $n = 2$ case, a correct analysis of the one-dimensional case).

0 points: Student does not make meaningful constructive progress towards the solution.

Note: Some students attempted a proof by induction. This is invalid (and should not not receive any points) since even if two sets of n points satisfy the key inequality in the problem statement, subsets of $n-1$ points do not necessarily satisfy this inequality. However, students who attempt a proof by induction can receive points for other constructive progress towards the solution.

Problem 5/3/28:

Note: Part (a) is worth **2 points** and part (b) and part (c) are collectively worth **3 points**.

Part (a): Award **2 points** for proving both the "right side implies left side" and "left side implies right side" implications. Award **1 point** if the student proves either direction, or the student has significant constructive progress for both directions.

Part (b) and part (c): Award points as follows:

Award **3 points** if the student solves both parts correctly.

Award **2 points** if the student solves one part correctly, and has acceptable constructive progress towards a solution to the other part. In part (b), an example of acceptable constructive progress would be proposing a correct (and specific) infinite set of positive numbers N that cannot be written as the sum of two elements of S (with at least some justification). In part (c), an example of acceptable constructive progress would be proposing a correct way of generating a specific infinite set of positive integers N that can be written as the sum of two elements of S (with at least some justification). A score of **2 points** should also be awarded if the student has nearly complete solutions to both parts (b) and (c) (i.e., each solution is complete and correct except for a minor flaw).

Award **1 point** if the student has acceptable constructive progress (as described above) on either part (b) or part (c) or on both parts, or if the student completely solves either part (b) or part (c) but does not have acceptable constructive progress on the other part.

Award **0 points** if the student does not make acceptable constructive progress (as described above).