USA Mathematical Talent Search

Round 1 Grading Rubric

Year 27 — Academic Year 2015–2016

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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct one additional point for a solution that is poorly written and/or hard to follow.

IMPORTANT NOTE: Appropriate credit should be awarded for full and partial solutions that use other correct approaches to the problem. Any solutions relying on computer methods should include the source code or specify the function call(s) (with arguments) used in a computer algebra system. Merely citing the name of a software package is not sufficient justification.

Problem 1/1/27:

Award **5 points** for the correct configuration of numbers. No justification is required. Withhold **1 point** for each incorrect entry.

Problem 2/1/27:

Student solution methods varied considerably. Any correct and complete solution should receive a score of **5 points**.

Students who used the method in the official solution should receive points as follows:

1 point: Student shows that a, b, and c are the roots of the polynomial

$$f(x) = (x-a)(x-b)(x-c) = x^3 - dx^2 + 1110x - 1000,$$

where $d > 0$.

4 points: Student uses the preceding fact to show that $10 < c < 100$. These points are divided as follows: Award **1 point** if the student proposes a comparison to the polynomial $g(x) = (x-1)(x-10)(x-100) = x^3 - 111x^2 + 1110x - 1000$. Award **1 point** if the student shows (with justification) that

$$g(c) = g(c) - f(c) < 0.$$

Award **1 point** if the student identifies the range of values of c, namely $(0,1) \cup (10,100)$, for which $g(c) < 0$. Award **1 point** if the student shows (with justification) that since $g(c) < 0$ and $a < 1$ (equivalently, $bc > 1000$), $10 < c < 100$.

For students who try to show that a contradiction emerges if $c \leq 10$ or $c \geq 100$, award **1 point** for proposing a proof by contradiction. Award **2 points** if the student shows that $c \leq 10$ leads to a contradiction, and award **2 points** if the student shows that $c \geq 100$ leads to a contradiction. For each of these cases, award **1 point** if the student makes constructive progress or if the justification is incomplete.

Students who use other methods and obtain the following intermediate results should receive points as follows:

1 point: Student shows that since $a < 1$, $bc > 1000$.

2 points: Student shows (with justification) that $b + c < 110$. Award **1 point** if the student makes constructive progress towards this result or if the justification is incomplete.

2 points: Student shows (with justification) that since $bc > 1000$ and $b + c < 110$, $10 < c < 100$. Award **1 point** if the student makes constructive progress towards this result or if the justification is incomplete.

For significant constructive progress towards or beyond the above results, award **1 point**.

Problem 3/1/27:

1 point: Student recognizes that it is sufficient to show that if there is a balancing point Q that is not a vertex of P (i.e., Q is inside P or on an edge), then there exists another balancing point.

2 points: Student shows (with justification) that as Q moves along a line,

$$f(Q) = [QV_1V_2] - [QV_2V_3] + \ldots + (-1)^{n-1}[QV_nV_1]$$

changes linearly. Award **1 point** for constructive progress towards this result (e.g., showing that as Q moves along a line, a particular $[QV_iV_{i+1}]$ changes linearly) or if the justification is missing or incomplete.

2 points: Student explains why the fact that $f(Q)$ changes linearly as Q moves along a line entails that if Q is not a vertex, then P has additional balancing points. Award **1 point** for significant constructive progress towards this result or if the justification is incomplete.

Problem 4/1/27:

Note: Students' figures varied considerably. The guidelines below are based on the figures in the official solution. When grading, be sure to adjust the guidelines to match a particular student's figure.

1 point: Student obtains the correct answer of $2^7 = 128$ possible teams. Award this point even if no justification is provided. If the student obtains the answer of 127 possible teams (omitting the team with 0 members), they should receive this point if and only if at least minimal justification is provided. Students should not receive full credit **(5 points)** on this problem if they leave out the possible team that has 0 members. **Exception**: If the student explicitly says that it is impossible to have a team with 0 members, the student should receive full credit if the solution is otherwise correct.

2 points: Student successfully uses requirements (i) and (ii) to show that if a player with $(\text{weight}, \text{height}) = (a, b)$ is on the team, then there can be no player with weight $\geq b$ on the team. Award **1 point** if the justification of this claim is missing or incomplete, or if the student provides a meaningful explanation of how at least one of the two requirements affects how the team can be constructed in a way that goes beyond simply restating the requirement(s). A graphical interpretation of one of the requirements is sufficient to receive this point.

2 points: Student uses combinatorics (with justification) to provide an accurate description of the set of all possible valid paths. To receive these points, the student does **NOT** need to mention the possible team that has 0 players or calculate the final numerical answer. Award **1 point** if the justification is missing or incomplete. Award **1 point** of partial credit if the student creates a valid path from p_0 (the player with the shortest height and lowest weight) to a point on the line $y = 197$ that satisfies all constraints (i.e., requirements (i) and (ii), and each step is up or to the right) or otherwise makes significant constructive progress.

Note: Students who did not use a "paths" interpretation, but who came up with a correct combinatorics (or alternative) argument including the correct answer should receive full credit **(5 points)**. Appropriate partial credit should be awarded for incomplete or partially correct solutions based on the extent of the student's progress and the above guidelines.

Problem 5/1/27:

1 point: Student obtains the correct answer that $n = 6m$, where m is a positive integer (no justification required). To receive a total of **1 point**, it is sufficient for a student to provide a specific example that works (e.g., $6 = 1 + 2 + 3$).

1 point: Student proves that $n = 6m$ works for all positive integers m.

3 points: Student proves that $n \neq 6m$ does not work. Partial credit should be awarded for significant constructive progress towards this result.

Note: Student solutions varied considerably, and as always, full credit should be awarded for other correct approaches, and appropriate partial credit should be awarded for incomplete or partially correct solutions based on the extent of the student's constructive progress.