# USA Mathematical Talent Search

Round 2 Grading Criteria

Year 22 — Academic Year 2010–2011

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**IMPORTANT NOTE:** On **all** problems, the graders have the discretion to deduct 1 additional point for a solution that is poorly written.

**Problem 1/2/22:**

On this problem award 3 points for existence and 2 points for uniqueness. This means any correctly filled grid is automatically worth at least 3 points. Any data implying how the grid could be filled (without the filled grid) must be accompanied by a proof that the set of entries satisfies all the necessary properties.

Most uniqueness proofs rely on generating a small number of cases and then eliminating all but the solution. Give one point for significant progress and two points for a clear and complete proof.

Some students may have coded this problem. Award three points for a program that shows existence (explicitily) and another point for uniqueness. Award the remaining 1 points for the explanation and mathematical justification of the program.

**Problem 2/2/22:**

Part (a) is worth 3 points and part (b) is worth 2 points. The statement that the shortest sequence has 5 terms, with any attempt at justification is worth one point. The statement that there are four 5-term sequences with some minimal attempt at justification is worth an additional 1 point.

On Part (a) award 1 point for any solution that exhibits a length 5 tworiffic sequence. Award 2 points to any proof that 5 is minimal having a minor flaw. Award 3 points to a correct and complete proof.

On Part (b) award 1 point for a solution exhibiting or implying the four possible solutions and 2 points for a clear and complete solution. Values other than 4 are worth at most one point and only if the argument shows significant insight into the problem.

**Problem 3/2/22:**

Give 1 point for the answer 7/12. Also always give at least one point for observing that, by symmetry, any path from the vertex opposite the Gortha to the Gortha ends with Richard feeding the Gortha exactly 1/3 of the time. On the other hand, treat any solution that gives an answer of less than 1/3 for Richard feeding the Gortha more harshly. Such a solution can earn at most 3 points and only in the most extreme cases.

Matrix solutions are fine. Computer aided matrix computations are also acceptable. Any computer-aided solution must describe how (why) the system is set up and then explain what method was used to solve it. (Grade these based entirely on how well the student formally sets up the problem and interprets the result.) Almost-integer arithmetic is fine (since the denominators are clearly bounded) *for this problem only*: if a computer returns 0.58333 and the student replaces this with 7/12, accept this.

Students might also use sophisticated tools for this problem. This is fine as long as the student justifies the tool and the method of application is explained.

Several students try to solve this with geometric series. It's not clear how such a solution could work without using matrices or somehow diagonalizing the system. Such matrix series solutions exist and are, of course, eligible for all 5 points.

- Give 1 point for 7/12 with little or no justification.
- Give 3 points to any student who sets up a correct system but fails to solve this system.
- Give 3 points to any almost-complete solution with a minor arithmetic mistake giving an implausible solution.
- Give 4 points to any almost-complete solution with a minor arithmetic mistake giving a plausible solution.
- Give 5 points to a clear, correct, and complete solution.

**Problem 4/2/22:**

This problem has many solutions. There are also many subtle assumptions that can be made that simplify diagrams. Be on the lookout for these (though we do allow the $B \leftrightarrow C$ symmetry to be used without justification). One common example is the assumption that $\overline{AD}$ intersects $\overline{BC}$ on the $A$ side of $\overline{BC}$.

- Give 1 point for a blind attack which shows some insight into the structure of the problem.
- Give 3 points to a clear plan with a weak (but fixable) error.
- Give 4 points for any "edge case failures." For example if the student makes a unjustified simplifying assumption (as in the intersection above), but the simplifying assumption is minor and the major points of the argument are present.
- Give 4 points for an almost correct proof with any other minor flaw.
- Give 5 points for a clear and correct proof.

**Problem 5/2/22:**

There are several aspects to this problem and many subtle issues with the proof. A student must have the correct answer, the correct algorithm, and a proof that 1008 is the minimum guaranteed time. There are many potential pitfalls in the proof. A student's proof of minimality must address all of these. Some of the more subtle points are:

- The argument must consider all games simultaneously, meaning any sequence of choices Ada could make will result in a victory for Zara in at most the proposed bound.
- The greedy algorithm might fail, globally or locally. As a simple example, the greedy approach argues for 1006 as the best first move, but does the student's solution clearly show (imply) that there is not a better combination of two moves where the first guess is not 1006?
- Zara might be able to chase more efficiently in the "endgame" by approaching the "opening" differently. For example, if Zara knows Ada must be at one of the two values 40 or 1000, then choosing 1000 *drastically* improves the game's state. Any solution must be able to deal with this case, which almost certainly means explicitly or implicitly eliminating it as a possiblity (see the interval construction from the official solutions).

There are two distinct ways to come by the number 1008. First, this is 1006 plus an overhead of two which Zara needs in order to locate Ada's number. Second, the extra two turns can be viewed as a the measure of Ada's leeway to be reticent. This second leads to a different proof that 1008 is minimal, by noting Ada can be at 1006 on the second turn, so cannot possibly be guaranteed to be cornered until turn 1008.

- A 4 should be awarded to a student who has the correct value and whose solution exhibits all of the nuances of the problem but has one or two small subtle holes.
- Give a score of 3 to any student with the correct algorithm, the correct value, and an attempt to prove that the value is minimal.
- Give a score of 2 to any student with the correct value and correct algorithm with no solid attempt to prove minimality.
- Give a score of 1 to any student with the correct value.

Subtract 1 point for any solution which fits the above rubric but miscalculates the answer (for example, a student who says 1007 instead of 1008).

**Problem 6/2/22:**

The answer to this question is yes. However a student who says "yes" with no valid justification should receive 0 points. Give any student who says "no" at most 3 points and only in the extreme case that the student shows almost complete understanding of the system. Any solution with a valid, explicit sequence (not just an algorithm for determining the sequence) must be awarded a 5. For example the last line of the first official solution, with a sentence to explain the notation, would be a 5 point solution.

There are two common approaches to this problem: constructive and descent (the official solutions contain one of each).

A constructive solution should be checked by the reader and if the sequence is correct a 5 is awarded. A minor flaw with a valid, justified approach should earn 3 points. Progress showing some insight into the problem, but giving an incorrect solution is worth 1 point. An invalid sequence with little or no justification should earn 0.

Almost any algorithmic approach will be via some sort of descent. The most common is to construct moves $n \mapsto \frac{1}{n-1}$ and $\frac{1}{n} \mapsto n-1$ and argue by descent. Notice that 2011 and 2 have different parity so be sure the student doesn't reach $\frac{1}{2}$ instead of 2. Such an approach, if correct, will describe the algorithm's steps and cite descent (induction is fine, as is just implying the sequence). Such a solution is worth 5 points. An arithmetic error in the algorithmic step, including reaching $\frac{1}{2}$ instead of 2, should reduce the score to at most 2 points.