USA Mathematical Talent Search
Round 1 Grading Criteria
Year 22 — Academic Year 2010–2011
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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct 1 additional point for a solution that is poorly written.

Problem 1/1/22:

Part (a) is worth 2 points and part (b) is worth 3 points. Incorrect answers for Part (a) which are a result of mistakes in setting up the count score 0 points.

Grading criteria on part (b):

- 1/3: A correct configuration with no supporting documentation that the configuration works.
- 2/3: A correct configuration with a poor explanation that the configuration works. An example of this is when a student creates a configuration by moving points from the configuration in part (a), and argues that n lines are eliminated and $n + 5$ lines are created, without proof that there aren't more than $n + 5$ new lines created. Another example is a cube projection solution that does not acknowledge that some projection orientations cannot be used.
- 3/3: Correct configuration with proof. We give 3 points for the cube projection solution provided that the student acknowledge that there are some orientations that must be avoided.

Problem 2/1/22:

Some students took the approach of drawing the entire path of the beam within one triangle, while others took the approach of reflecting the entire triangle over and over, and treating the path of the beam as a straight line. In either case, points are allocated as follows:

- Finding the correct path of the beam: 1 point.
- Proving that this is the correct path: 3 points.
- Finding the correct angle and number of reflections: 1 point. (Note: Some students gave 6 as the number of reflections, probably counting the number of legs of the beam's path. Don't deduct for this minor error if the solution is otherwise absolutely clear.)

Important points to watch out for:

- Students reflecting the whole triangle must prove that the path of the beam hits the sides they claim it hits. This usually involves identifying the reflex angle at B, for example.
- Be strict on rigor; many students make unsubstantiated claims about the beam's path (or about what paths are impossible).

- Many students who attacked the problem by following the beam within the original triangle did so by successively showing that it was impossible to return to A with 1, 2, 3, or 4 reflections. Some of these students dismiss the 4 reflection case by only analyzing the symmetric case. This is incomplete—their solution must also show that there are not two 4-reflection possibilities that are mirror images. This error should cost the student 1 point.
- Solutions that gave the correct 5-reflection answer and rigorously calculated the correct angle, but that made no argument about the minimum number of reflections earned only 2 points.
- Miscaluclating the angles of reflection lost one point if the error gave only a wrong value or two points if the error allowed an impossible reflection.

Problem 3/1/22:

Any solution which does not give an exact and correct answer is worth at most 4 points.

Several students were able to squeeze this problem through a computer numerically. Give such an approximation at most 4 points, and only if the solution clearly proves c is unique, checks the given equation $\left(1 = \frac{1}{r^2+s^2} + \cdots\right)$, and (this should follow trivially) implies that arbitrary precision can be achieved.

A student who uses software to manipulate the expressions entirely symbolically may earn 5 points for a solution which can be verified. Such a student must explicitly control any necessary casework and properly justify the system of nonlinear equations fed into the program.

- Give at least 1 point to any student who attempts to use Vieta.
- Give 2 points for moderate progress.
- Give 4 points for an essentially correct solution with any algebraic mishaps (including missing $c > 0$).
- Give 5 points for a complete and clear solution.

Problem 4/1/22:

This problem has two parts. The student must show that the two choices of tangent (closer to C than D, or vice versa) produce circles of the same radius, and must show that this radius is independent of the location of the original two circles. There are many approaches, including a variety of synthetic, analytic, and trigonometric solutions.

- 1/5: The student produced at least some valid geometric observations in the direction of a solution.
- 2/5: Student proves that the two choices of tangent produce circles of the same radius, but does not show that this radius is independent of the location of Rebecca's and Sasha's circles.
- 3/5: The student shows that the circle for one choice of tangent is independent of the location of Rebecca's and Sasha's circles. Note that some solution methods cover both configurations simultaneously.
- 4/5: Student's method covers both configuration simultaneously, but student does not acknowledge that both configurations must be considered. Or, a minor step is left out of an otherwise correct solution.
- 5/5: Correct and justified solution.

Problem 5/1/22:

- Give one point for any student who exhibits the peculiar triangle.
- Give two points to any student who finds that there can be only one triangle and one quadrilateral up to congruence.
- Give five points to any student with a clear and correct answer.

Partial credit for proving that there are no other peculiar polygons is at the reader's discretion. The crux of the argument is that once there are four vertices there is no way to add a fifth without creating duplicates. Most of the sensitivity of the grading should focus on this step. Most techniques address the pentagon as a specific case.

Problem 6/1/22:

This problem has 2 parts. Give 4 points for a complete solution to the first part and 1 point for the computation in the second part.

The first part should contain an induction argument. The standard induction has two key components that need to be made clear for a complete solution.

- We assume that the players after player n play optimally and we make no assumptions about the players before player n.
- We must prove that if player n does not play optimally he will always get a worse present. (Swapping when he shouldn't is trivial and might be omitted, but the student must explicitly argue that if there is a large present available to steal and he does not steal it then he will get a present that is less valuable.)

Any solution which deviates from this type of induction should be graded on a case-by-case basis.

On part (a) give:

- 0/4 for a faulty strategy with little justification.
- 1/4 for a faulty strategy where the justification shows significant insight into the problem.
- 1/4 for the correct strategy with little to no justification.
- 2/4 for the correct strategy with purely empirical justification which shows some insight into the key strategy.
- 3/4 for the correct strategy with a reasonable but flawed attempt at proof (a flawed attempt at the induction, for example).
- 4/4 for a correct strategy and complete proof.

For part (b) give 1/1 if the student has the correct solution. Accept decimal approximations for this part of this problem only, assuming that there is enough work shown to generate the actual solution.

Notice that a student could potentially extrapolate from small cases and guess the solution to part (b). This is worth at least 1 point (for part (b) only) and possibly up to 3 points if the empirical work also produces the correct strategy and some understanding of the meaning of the correct strategy.