USA Mathematical Talent Search

Round 3 Grading Criteria

Year 21 — Academic Year 2009–2010

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IMPORTANT NOTE: On **all** problems, the graders have the discretion to deduct 1 additional point for a solution that is poorly written.

Problem 1/3/21:

Give 0/5 for any solution which begins with an unwarranted assumption which alters/simplifies the problem.

Give 1/5 for a reasonable start, such as finding the equations relating the distance from point P to a side with the distances from point P to the endpoints of that side.

Give 2/5 for any solution which assumes a value for an unknown length on the figure, and uses that length in the calculations to get the correct answer (for example, assuming that some distances must be integers).

Any solution which finds the formula $a^2b^2 = h^2(a^2 + b^2)$ is worth at least 3 points.

Give 4/5 for a rigorous solution with minor arithmetic or algebraic error.

Give 5/5 for any rigorous calculation without unfounded assumptions that reached the correct answer.

Problem 2/3/21:

Proof that the student's answer makes the expression an integer, either explicitly or implicitly, is worth 1 point.

The other four points are given for the value and a proof that this value works. Any solution which gives a **valid** answer with some attempt at justification is worth at least 2/4 points. A correct tactic with a minor flaw is worth 3/4.

As a specific case, assign 3/5 for a method for constructing n that looks valid and provable, shows the output is an integer (explicitly or implicitly) but the proof of indivisibility given is flawed.

Note: 500! is an incorrect answer. Students can still get 1 point for proving that it produces an integer, and up to an additional 2 points for significant work in their attempt to show that it has no prime factor less than 500. In particular, a solution that follows the model of the official solution, but which uses 500! (for example) instead of $(500!)^2$, might receive as much as 3 points total if everything is done correctly before the final step.

Problem 3/3/21:

Give a maximum of 1 point for any solution which begins by arranging the squares.

Give a maximum of 3 points for any solution that does not get both 24×19 and $3 + \frac{\pi}{4}$.

Points on this problem are broken down as follows:

2 points for realizing that the center must lie in the 24×19 region.

2 points for the neighborhood of a square (1 for understanding the region, 1 for computing the area correctly).

1 point for clear and valid completion.

Problem 4/3/21:

The grading for this problem is broken into two stages:

Assign three points to the equation $2009 = (a-1)(b-1)/2$. Full credit goes to any solution with a valid proof or citation. Students may cite a relevant result, such as the Frobenius Coin Problem or the Chicken McNugget Theorem. However, the result necessary for this problem is actually a generalization of those results (which in common usage only deal with the maximum nonattainable number, not the total number of nonattainable numbers), so implying or claiming (without proof or further citation) that the result in this problem follows from either cited source is worth at most 2 points. Stating, and then legitimately citing or proving, the necessary generalization is worth the full 3 points.

Assign the remaining 2 points for a clean and clear proof that $50 + 83$ is the *unique* solution. This can be either via machinery or tenacity. Uniqueness is required for a complete proof.

Computer-aided proofs are acceptable. A computer proof must contain the following elements: a rigorous proof that every possible pair of positive integers, (a, b), has been checked, and a rigorous proof that the program has the correct count for the size of the set of non-expressible integers (specifically a *valid* condition for termination of the loop). Assign 1 point for a program with no flaws, and the remaining 4 points for proofs that the program runs correctly. At most one point should be awarded to most computer-aided solutions which rely on the assumption that the solution is unique.

Problem 5/3/21:

Give at least 1/5 for any solution which proves $a_n + b_n + c_n = n + 1$.

Give 2/5 to any solution which attempts to use cube roots of unity in a reasonable way. For example, the expression $a_n + \omega b_n + \omega^2 c_n$ with *any* justification is automatically worth at least 2 points.

Give 3/5 for any solution which finds a usable closed form, but fails to prove the inequality, such as $\prod \left(1 + \frac{\omega}{k}\right)$ or $\prod \frac{k^2-k+1}{k^2}$. This should be a rare score, resulting only from a failed induction, or a solution which stops at this point.

Give 4/5 for a valid proof with a minor flaw.

Give 5/5 for a clear and complete proof.

Many of the induction arguments possible on this problem fail in subtle ways, so these must be graded very closely. An induction argument where the induction step is not treated explicitly will be assumed false.