USA Mathematical Talent Search

Round 2 Problems

Year 36 — Academic Year 2024-2025

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **December 2, 2024** via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on December 2, 2024.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 503
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before December 2, 2024.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to `www.usamts.org` and visiting the "Account" page.

6. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My Scores". You will also receive an email when your scores and comments are available (provided that you did item #5 above).

These are only part of the complete rules.
Please read the entire rules at `www.usamts.org`.

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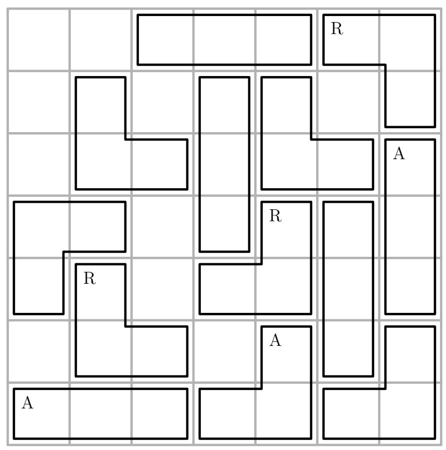
Each problem is worth 5 points.

1/2/36. Fill each cell with an integer from 1-7 so each number appears exactly once in each row and column. In each “cage” of three cells, the three numbers must be valid lengths for the sides of a non-degenerate triangle. Additionally, if a cage has an “A”, the triangle must be acute, and if the cage has an “R”, the triangle must be right.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/36. In how many ways can a 3×3 grid be filled with integers from 1 to 12 such that all three of the following conditions are satisfied: (a) both 1 and 2 appear in the grid, (b) the grid contains at most 8 distinct values, and (c) the sums of the numbers in each row, each column, and both main diagonals are all the same? Rotations and reflections are considered the same.

3/2/36. $\triangle ABC$ is an equilateral triangle. D is a point on $\overline{AC}$, and E is a point on $\overline{BD}$. Let P and Q be the circumcenters of $\triangle ABD$ and $\triangle AED$, respectively. Prove that $\triangle EPQ$ is an equilateral triangle if and only if $\overline{AB} \perp \overline{CE}$.

(The problems are continued on the next page.)

4/2/36. Let $x_1 < x_2 < \cdots < x_n$ (with $n \geq 2$) and let S be the set of all the x_i. Let T be a randomly chosen subset of S. What is the expected value of the indexed alternating sum of T? Express your answer in terms of the x_i.

Note: We define the indexed alternating sum of T as

$$\sum_{i=1}^{|T|} (-1)^{i+1}(i)T[i],$$

where $T[i]$ is the ith element of T when listed in increasing order. For example, if $T = \{1, 3, 5\}$ then the indexed alternating sum of T is

$$1 \cdot 1 - 2 \cdot 3 + 3 \cdot 5 = 10.$$

Alternating sums of empty sets are defined to be 0.

5/2/36. Prove that there is no polynomial $P(x)$ with integer coefficients such that

$$P(\sqrt[3]{5} + \sqrt[3]{25}) = 2\sqrt[3]{5} + 3\sqrt[3]{25}.$$

Problems by Ivan Dosev, Nairit Sarkar, and USAMTS Staff.
Round 2 Solutions must be submitted by **December 2, 2024**.
Please visit **https://www.usamts.org** for details about solution submission.