USA Mathematical Talent Search

Round 1 Problems

Year 36 — Academic Year 2024-2025

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **October 15, 2024** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on October 15, 2024.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 503
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before October 15, 2024.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "Account" page. If you registered for last year's contest, you need to click the "Update Profile" button to re-register for this year's contest.

6. Round 1 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My Scores". You will also receive an email when your scores and comments are available (provided that you did item #5 above).

These are only part of the complete rules.
Please read the entire rules at www.usamts.org.

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Each problem is worth 5 points.

1/1/36. The “Manhattan distance” between two cells is the shortest distance between those cells when traveling up, down, left, or right, as if one were traveling along city blocks rather than as the crow flies.

Place numbers from 1-6 in some cells so the following criteria are satisfied:

1. A cell contains at most one number. Cells can be left empty.
2. For each cell containing a number N in the grid, exactly two other cells containing N are at a Manhattan distance of N.
3. For each cell containing a number N in the grid, no other cells containing N are at a Manhattan distance less than N.

		4					
	5	2				3	
3			1				
				1			4
	2				6	5	
					4		

There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/1/36. A regular hexagon is placed on top of a unit circle such that one vertex coincides with the center of the circle, exactly two vertices lie on the circumference of the circle, and exactly one vertex lies outside of the circle. Determine the area of the hexagon.

(The problems are continued on the next page.)

3/1/36. A sequence of integers $x_1, x_2, \ldots, x_k$ is called *fibtastic* if the difference between any two consecutive elements in the sequence is a Fibonacci number.

The integers from 1 to 2024 are split into two groups, each written in increasing order. Group A is $a_1, a_2, \ldots, a_m$ and Group B is $b_1, b_2, \ldots, b_n$.

Find the largest integer M such that we can guarantee that we can pick M consecutive elements from either Group A or Group B which form a fibtastic sequence.

As an illustrative example, if a group of numbers is $2, 4, 11, 12, 13, 16, 18, 27, 29, 30$, the longest fibtastic sequence is $11, 12, 13, 16, 18$, which has length 5.

Note: We've received questions about what is meant by "Find the largest integer M such that we can guarantee ..." We mean "guarantee" in the sense that if we distribute 9 bananas to 2 monkeys, some monkey is guaranteed to get at least 5 bananas regardless of how the bananas are distributed, even though the other monkey will get fewer than 5 bananas.

4/1/36. During a lecture, each of 26 mathematicians falls asleep exactly once, and stays asleep for a nonzero amount of time. Each mathematician is awake at the moment the lecture starts, and the moment the lecture finishes. Prove that there are either 6 mathematicians such that no two are asleep at the same time, or 6 mathematicians such that there is some point in time during which all 6 are asleep.

Note: We consider a mathematician to be asleep at the moment they fall asleep, and awake at the moment they wake up.

5/1/36. Let $f(x) = x^2 + bx + 1$ for some real number b. Across all possible values of b, find all possible values for the number of integers x that satisfy $f(f(x) + x) < 0$.

That is, if there are some values of b that give us 180 integer solutions for x and there are other values of b that give us 314 integer solutions for x (and these are the only possibilities), the answer would be $\boxed{180, 314}$.

Problems by Natalie Moeller, Carl Yerger, Tuan Nguen, and USAMTS Staff.
Round 1 Solutions must be submitted by **October 15, 2024**.
Please visit **http://www.usamts.org** for details about solution submission.