USA Mathematical Talent Search

Round 3 Problems

Year 35 — Academic Year 2023-2024

www.usamts.org

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **January 3, 2024** via one (and only one!) of the methods below:

   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on January 3, 2024.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before January 3, 2024.

5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.

6. Round 3 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #5 above).

These are only part of the complete rules.
Please read the entire rules at www.usamts.org.

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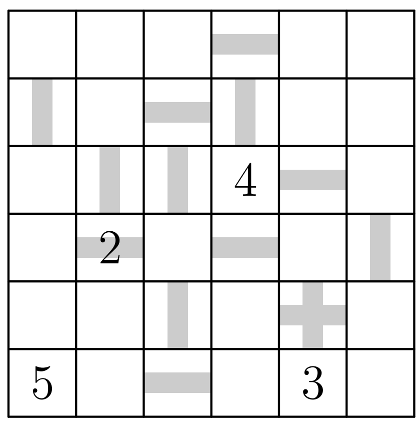
Each problem is worth 5 points.

1/3/35. Fill in the grid with the numbers 1 to 6 so that each number appears exactly once in each row and column. A horizontal gray line marks any cell when it is the middle cell of the three consecutive cells with the largest sum in that row. Similarly, a vertical gray line marks any cell when it is the middle of the three consecutive cells with the largest sum in that column. If there is a tie, multiple lines are drawn in the row or column. A cell can have both lines drawn, with the appearance of a plus sign.

For example, if a filled row had 1 5 4 2 3 6, horizontal lines would be drawn in the two cells with 4 and 3, since out of $1+5+4$, $5+4+2$, $4+2+3$, and $2+3+6$, the largest sums are $5+4+2$ and $2+3+6$.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/3/35. Grogg takes an $a \times b \times c$ rectangular block (where a, b, c are positive integers), paints the outside of it purple, and cuts it into abc small $1 \times 1 \times 1$ cubes. He then places all the small cubes into a bag, and Winnie reaches in and randomly picks one of the small cubes. If the probability that Winnie picks a totally unpainted cube is 20%, determine all possible values of the number of cubes in the bag.

(The problems are continued on the next page.)

3/3/35. Lizzie and Alex are playing a game on the whiteboard. Initially, n twos are written on the board. On a player's turn they must either

1. change any single positive number to 0, or
2. subtract one from any positive number of positive numbers on the board.

The game ends once all numbers are 0, and the last player who made a move wins. If Lizzie always plays first, find all n for which Lizzie has a winning strategy.

4/3/35. In this problem, a *simple polygon* is a polygon that does not intersect itself and has no holes, and a *side* of a polygon is a maximal set of collinear, consecutive line segments in the polygon. In particular, we allow two or more consecutive vertices in a simple polygon to be identical, and three or more consecutive vertices in a simple polygon to be collinear. By convention, polygons must have at least three sides. A simple polygon is *convex* if every one of its interior angles is 180° or less. A simple polygon is *concave* if it is not convex.

Let P be the plane. Prove or disprove each of the following statements:

(a) There exists a function $f : P \to P$ such that for all positive integers $n \geq 4$, if $v_1, v_2, \ldots, v_n$ are the vertices of a simple concave n-sided polygon in some order, then $f(v_1), f(v_2), \ldots, f(v_n)$ are the vertices of a simple convex polygon in some order (which may or may not have n sides).

(b) There exists a function $f : P \to P$ such that for all positive integers $n \geq 4$, if $v_1, v_2, \ldots, v_n$ are the vertices of a simple convex n-sided polygon in some order, then $f(v_1), f(v_2), \ldots, f(v_n)$ are the vertices of a simple concave polygon in some order (which may or may not have n sides).

5/3/35. Let ω be the unit circle in the xy-plane in 3-dimensional space. Find all points P not on the xy-plane that satisfy the following condition: There exist points A, B, and C on ω such that

$$\angle APB = \angle APC = \angle BPC = 90^\circ.$$

Problems by USAMTS Staff.
Round 3 Solutions must be submitted by **January 3, 2024**.
Please visit **http://www.usamts.org** for details about solution submission.