USA Mathematical Talent Search

Round 2 Problems

Year 35 — Academic Year 2023-2024

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Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name, username, and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
4. Submit your solutions by **November 27, 2023** via one (and only one!) of the methods below:
   (a) Web: Log on to www.usamts.org to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on November 27, 2023.
   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before November 27, 2023.
5. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to www.usamts.org and visiting the "My USAMTS" pages.
6. Round 2 results will be posted at www.usamts.org when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #5 above).

These are only part of the complete rules.
Please read the entire rules at www.usamts.org.

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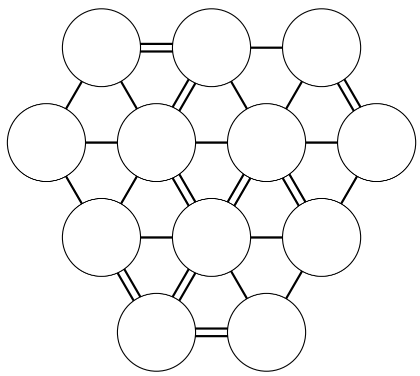
Each problem is worth 5 points.

1/2/35. In the diagram below, fill the 12 circles with numbers from the following bank so that each number is used once. Two circles connected by a single line must contain relatively prime numbers. Two circles connected by a double line must contain numbers that are not relatively prime.

Bank: 20, 21, 22, 23, 24, 25, 27, 28, 30, 32, 33, 35



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/35. Malmer Pebane's apartment uses a six-digit access code, with leading zeros allowed. He noticed that his fingers leave smudges that reveal which digits were pressed. He decided to change his access code to provide the largest number of possible combinations for a burglar to try when the digits are known. For each number of distinct digits that could be used in the access code, calculate the number of possible combinations when the digits are known but their order and frequency are not known. For example, if there are smudges on 3 and 9, two possible codes are 393939 and 993999. Which number of distinct digits in the access code offers the most combinations?

(The problems are continued on the next page.)

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3/2/35. We say that three numbers are *balanced* if either all three numbers are the same, or they are all different. A grid consisting of hexagons is presented in Figure 1. Each hexagon is filled with the number 1, 2, or 3, so that for any three hexagons that are mutually adjacent and oriented with two hexagons on the bottom and one hexagon on the top (as in Figure 3), the three numbers in the hexagons are balanced. Prove that when the grid is filled completely, the three numbers in the three shaded hexagons are balanced.

(An example of a partially filled-in grid is shown in Figure 2. There are other ways of filling in the grid.)

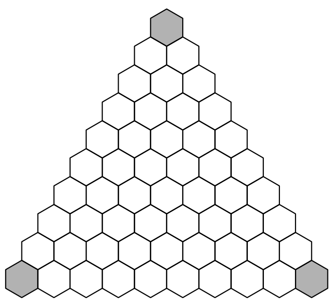

Figure 1

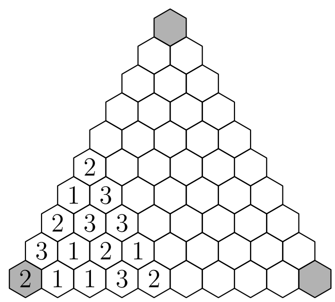

Figure 2

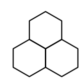

Figure 3

4/2/35. The incircle of triangle ABC with $AB \neq AC$ has center I and is tangent to BC, CA, and AB at D, E, and F respectively. The circumcircle of triangle ADI intersects AB and AC again at X and Y. Prove that EF bisects XY.

5/2/35. Let m and n be positive integers. Let S be the set of all points (x, y) with integer coordinates such that $1 \leq x, y \leq m+n-1$ and $m+1 \leq x+y \leq 2m+n-1$. Let L be the set of the $3m+3n-3$ lines parallel to one of $x=0$, $y=0$, or $x+y=0$ and passing through at least one point in S. For which pairs (m, n) does there exist a subset T of S such that every line in L intersects an odd number of elements of T?

Problems by Holden Mui and USAMTS Staff.

Round 2 Solutions must be submitted by **November 27, 2023**.

Please visit **http://www.usamts.org** for details about solution submission.