USA Mathematical Talent Search
Round 1 Problems
Year 35 — Academic Year 2023-2024
`www.usamts.org`

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.

2. Put your name, username, and USAMTS ID# on **every page you submit**.

3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.

4. Submit your solutions by **October 16, 2023** via one (and only one!) of the methods below:

   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on October 16, 2023.

   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before October 16, 2023.

5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.

6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to `www.usamts.org` and visiting the "My USAMTS" pages.

7. Round 1 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules at `www.usamts.org`.

USA Mathematical Talent Search

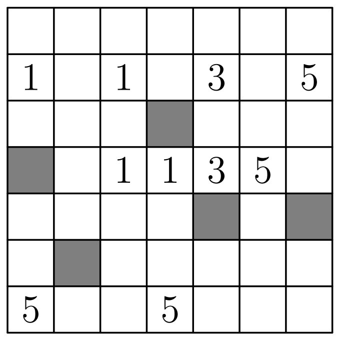
Round 1 Problems

Year 35 — Academic Year 2023-2024

www.usamts.org

Each problem is worth 5 points.

1/1/35. Fill each unshaded cell of the grid with a number that is either 1, 3, or 5. For each cell, exactly one of the touching cells must contain the same number. Here touching includes cells that only share a point, i.e. touch diagonally.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/1/35. Suppose that the 101 positive integers

$$2024,\ 2025,\ 2026,\ \ldots,\ 2124$$

are concatenated in some order to form a 404-digit number. Can this number be prime?

3/1/35. Let $n \geq 2$ be a positive integer, and suppose buildings of height $1, 2, \ldots, n$ are built in a row on a street. Two distinct buildings are said to be *roof-friendly* if every building between the two is shorter than both buildings in the pair. For example, if the buildings are arranged $5, 3, 6, 2, 1, 4$, there are 8 roof-friendly pairs: $(5, 3)$, $(5, 6)$, $(3, 6)$, $(6, 2)$, $(6, 4)$, $(2, 1)$, $(2, 4)$, $(1, 4)$. Find, with proof, the minimum and maximum possible number of roof-friendly pairs of buildings, in terms of n.

(The problems are continued on the next page.)

USA Mathematical Talent Search

Round 1 Problems

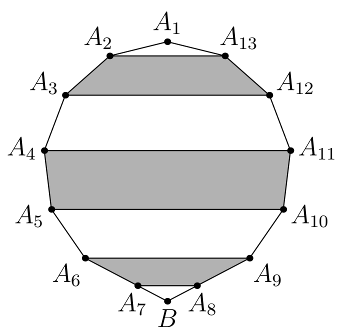
Year 35 — Academic Year 2023-2024

www.usamts.org

4/1/35. Prove that, for any real numbers $1 \le \sqrt{x} \le y \le x^2$, the following system of equations has a real solution (a, b, c):

$$a + b + c = \frac{x + x^2 + x^4 + y + y^2 + y^4}{2}$$
$$ab + ac + bc = \frac{x^3 + x^5 + x^6 + y^3 + y^5 + y^6}{2}$$
$$abc = \frac{x^7 + y^7}{2}.$$

5/1/35. Let $A_1A_2A_3 \cdots A_{13}$ be a regular 13-gon, and let lines A_6A_7 and A_8A_9 intersect at B. Show that the shaded area below is half the area of the entire polygon (including triangle A_7A_8B).



Problems by Ray Li and USAMTS Staff.

Round 1 Solutions must be submitted by **October 16, 2023**.

Please visit **http://www.usamts.org** for details about solution submission.