USAMTS

USA Mathematical Talent Search

Round 2 Problems

Year 34 — Academic Year 2022-2023

`www.usamts.org`

Important information:

1. **You must show your work and prove your answers on all problems.** If you just send a numerical answer with no proof for a problem other than Problem 1, you will get no more than 1 point.
2. Put your name, username, and USAMTS ID# on **every page you submit**.
3. No single page should contain solutions to more than one problem. Every solution you submit should begin on a new page.
4. Submit your solutions by **November 28, 2022** via one (and only one!) of the methods below:
   (a) Web: Log on to `www.usamts.org` to upload a PDF file containing your solutions. (No other file type will be accepted.)
   Deadline: 10 PM Eastern / 7 PM Pacific on November 28, 2022.
   (b) Mail: USAMTS
   55 Exchange Place
   Suite 603
   New York, NY 10005
   Deadline: Solutions must be postmarked on or before November 28, 2022.
5. Once you send in your solutions, that submission is final. You cannot resubmit solutions.
6. Confirm that your email address in your USAMTS Profile is correct. You can do so by logging on to `www.usamts.org` and visiting the "My USAMTS" pages.
7. Round 2 results will be posted at `www.usamts.org` when available. To see your results, log on to the USAMTS website, then go to "My USAMTS". You will also receive an email when your scores and comments are available (provided that you did item #6 above).

These are only part of the complete rules.
Please read the entire rules at `www.usamts.org`.

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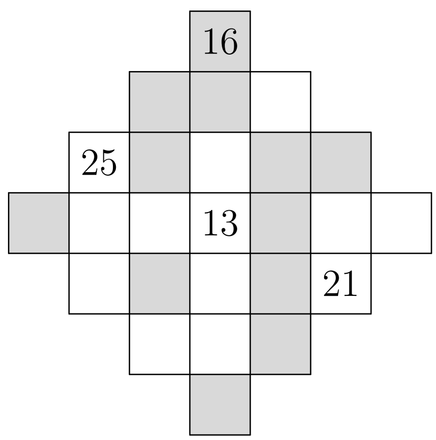
Each problem is worth 5 points.

1/2/34. Fill in the grid below with the numbers 1 through 25, with each number used exactly once, subject to the following constraints:

1. Each shaded square contains an even number, and each unshaded square contains an odd number.

2. For any pair of squares that share a side, if x and y are the two numbers in those squares, then either $x \geq 2y$ or $y \geq 2x$.

Four numbers have been filled in already.



There is a unique solution, but you do not need to prove that your answer is the only one possible. You merely need to find an answer that satisfies the conditions of the problem. (Note: In any other USAMTS problem, you need to provide a full proof. Only in this problem is an answer without justification acceptable.)

2/2/34. Grogg's favorite positive integer is $n \geq 2$, and Grogg has a lucky coin that comes up heads with some fixed probability p, where $0 < p < 1$. Once each day, Grogg flips his coin, and if it comes up heads, he does two things:

1. He eats a cookie.

2. He then flips the coin n more times. If the result of these n flips is $n-1$ heads and 1 tail (in any order), he eats another cookie.

He never eats a cookie except as a result of his coin flips. Find all possible values of n and p such that the expected value of the number of cookies that Grogg eats each day is exactly 1.

3/2/34. Let $n \geq 3$ be a positive integer. Alex and Lizzie play a game. Alex chooses n positive integers (not necessarily distinct), writes them on a blackboard, and does nothing further. Then, Lizzie is allowed to pick some of the numbers—but not all of them—and replace them each by their average. For example, if $n = 7$ and the numbers Alex writes on the blackboard to start are $1, 2, 4, 5, 9, 4, 11$, then on her first turn Lizzie could pick $1, 4, 9$, erase them, and replace them each with the number $\frac{1+4+9}{3}$, leaving on the blackboard the numbers $\frac{14}{3}, 2, \frac{14}{3}, 5, \frac{14}{3}, 4, 11$. Lizzie can repeat this process of selecting and averaging some numbers as often as she wants. Lizzie wins the game if eventually all of the numbers written on the blackboard are equal. Find all positive integers $n \geq 3$ such that no matter what numbers Alex picks, Lizzie can win the game.

4/2/34. A *lattice point* of the coordinate plane is a point (x, y) in which both x and y are integers. Let $k \geq 2$ be a positive integer. Find the smallest positive integer c_k (which may depend on k) such that every lattice point can be colored with one of c_k colors, subject to the following two conditions:

1. If (x, y) and (a, b) are two distinct neighboring points; that is, $|x-a| \leq 1$ and $|y-b| \leq 1$, then (x, y) and (a, b) must be different colors.

2. If (x, y) and (a, b) are two lattice points such that $x \equiv a \pmod{k}$ and $y \equiv b \pmod{k}$, then (x, y) and (a, b) must be the same color.

5/2/34. Let r and s be positive real numbers, and let $A = (0, 0)$, $B = (1, 0)$, $C = (r, s)$, and $D = (r + 1, s)$ be points on the coordinate plane. Find a point P, whose coordinates are expressed in terms of r and s, with the following property: if E is any point on the interior of line segment $\overline{AB}$, and F is the unique point other than E that lies on the circumcircles of triangles BCE and ADE, then P lies on line $\overleftrightarrow{EF}$.

Problems by Agustin Marchionna, Andrew Wu, and USAMTS Staff.
Round 2 Solutions must be submitted by **November 28, 2022**.
Please visit **http://www.usamts.org** for details about solution submission.